Analysis for Mixtures of Fluids

Dissertation

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch–Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich–Wilhelms–Universität Bonn

vorgelegt von

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aus

Duisburg

Bonn 2005

Angefertigt mit Genehmigung der Mathematisch–Naturwissenschaftlichen Fakultät
der Rheinischen Friedrich–Wilhelms–Universität Bonn

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Tag der Promotion: 21.3.2005

Abstract

In this thesis, we are dealing with a model for mixtures of compressible fluids. We consider a system of Navier–Stokes-like equations which are coupled in the elliptic principal part as well as in the partial pressures.

The main assumption of this model is the principle of co-occupancy, i.e. at each point of the space which is occupied by the mixture there are particles belonging to each of the constituents. One defines for every component a density function, a velocity field and other physical quantities. For isothermal flows considered here, the basic balance laws as the conservation of mass and the balance of momentum are stated separately for each of the components, and the coupling between these equations takes place via shear force interactions, the pressure law and interaction terms.

This thesis aims at contributing to the mathematical theory for this set of equations, which is not very developed in more than one space dimension.

We address the following problems:

Firstly, we deal with a Stokes-like system with a linear pressure law in a bounded domain $\Omega \subset \mathbb{R}^3$. We prove the existence of weak solutions, regularity properties of the solutions and the strong convergence of approximate densities. These are the first results for the Stokes problem for mixtures in bounded domains.

Secondly, we show the compactness of solutions to the steady mixture model taking into account also the convective terms under the assumption of suitable estimates.

Finally, we present new methods for obtaining estimates:

On the one hand, we deal with the mixture model with convective terms in the steady case with a pressure behaving like $|\rho|^\gamma, \frac{5}{4} < \gamma \leq 5$. Under the assumption that we have solutions to the momentum equation we prove new L^p-estimates for the terms $\rho_i|u^{(i)}|^2$ and the densities ρ_i.

On the other hand, we prove a new exponential estimate for the densities in the case of the Stokes problem with a pressure law behaving like $|\rho|^\gamma, \gamma \geq 2$.

Contents

Chapter 1

Introduction

Nature provides countless examples of mixtures – e.g. in astrophysics, biology or geology. Most bodies are mixtures of two or more components, such as plasma or mixtures of gases surrounding celestial bodies, blood and biological tissues, suspensions, soil or porous rock penetrated by water and oil.

In this thesis, we consider mixtures of compressible fluids, where the expression '*fluid*' can stand for a *liquid* or a *gas*.

There are various approaches to model the behavior of mixtures of fluids. In this thesis we deal with a continuum mechanics model which is presented e.g. in the book by Rajagopal and Tao ([RT95], cf. also [Raj96]). Its origins date back to the pioneering works by Fick in 1855 ([Fic55]) and Darcy in 1856 ([Dar56]). A firm mathematical footing, which marks the beginning of the modern phase of the theory of mixtures, was provided by Truesdell in 1957 ([Tru57], cf. also [Tru84]). For a summary of the historical development in modelling mixtures of fluids in continuum mechanics see [AC76].

The basic assumption of this model is the principle of *co-occupancy*: At each point of the space which is occupied by the mixture there are at any time simultaneously particles belonging to each of the constituents.

This assumption is reasonable if the different components of the (real) mixture are sufficiently well densly distributed throughout the whole mixture. Then, within the context of an appropriate homogenization, each of the constituents can be viewed as a single continuum of its own right.

As the mixture deforms, each of these continua moves relative to each other.

Since we regard each of the constituents as a continuum of its own right, we can define for each of the components a density function, a velocity field, a partial stress tensor and other physical quantities.

The constituents obey the basic balance laws as the conservation of mass and the balance of momentum. For isothermal flows considered in this thesis, these equations are stated separately for each of the components. The coupling between the equations takes place via shear force interactions, the pressure law and interaction terms, which model for instance drag (in the equations for the balance of momentum) or chemical reactions (in the mass balance equations). One can say that the shear forces model the interactions inside one component, and the interaction terms model the interactions between the different components, so to say on the *boundary* between one constituent and another.

For reasons of lucidity, we ‘restrict’ ourselves in this thesis to the case of two components. From the point of view of mathematics, mixtures with N constituents, $N > 2$, can be treated completely analogously.

Let ρ_i be the density, $u^{(i)} = (u_1^{(i)}, u_2^{(i)}, u_3^{(i)})^T$ the velocity field for the ith component of the mixture, $i = 1, 2$. We use the notation

$$\rho = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}, \quad u = \begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix} = \begin{pmatrix} \left(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}\right)^T \\ \left(u_1^{(2)}, u_2^{(2)}, u_3^{(2)}\right)^T \end{pmatrix}.$$

Moreover, we denote by $T^{(i)}, f^{(i)}, J^{(i)}$ the Cauchy stress tensor, the density of the external forces and the momentum source (frequently also called interaction term) for the ith constituent.

We consider mixtures of fluids at constant temperature and we do not take into account chemical reactions.

Then the conservation of mass for the ith species reads

$$(\rho_i)_t + \operatorname{div}\left(\rho_i u^{(i)}\right) = 0\,, \tag{1.1}$$

and the balance of linear momentum for the ith component leads to

$$\left(\rho_i u^{(i)}\right)_t + \operatorname{div}\left(\rho_i u^{(i)} \otimes u^{(i)}\right) = \operatorname{div} T^{(i)} + \rho_i f^{(i)} + J^{(i)}\,. \tag{1.2}$$

Here and in the following, no summation convention over repeated indices is used unless explicitely mentioned.

Newton’s third law of motion “*For every action, there is an equal and opposite reaction*” implies that

$$J^{(2)} = -J^{(1)}\,.$$

For identifying the constitutive equations for the Cauchy stress tensor $T^{(i)}$ and the interaction terms $J^{(i)}$ we use the balance of the entropy for the whole mixture and the second law of thermodynamics. With ψ_i denoting the Helmholtz potential for the ith component, it reads

$$\sum_{i=1}^{2} T^{(i)} : \nabla u^{(i)} - J^{(1)} \cdot \left(u^{(1)} - u^{(2)}\right) - \sum_{i=1}^{2} \left[\rho_i \left(\psi_i\right)_t + \rho_i u^{(i)} \cdot \nabla \psi_i\right] \geq 0\,. \tag{1.3}$$

We assume that the energy-storage mechanism is the same for each species (up to a constant positive factor), i.e.

$$\psi_1 = c\psi_2\,,$$

and for $i = 1, 2$

$$\psi_i = c_i \Psi(c_1\rho_1 + c_2\rho_2)\,, \quad c_i > 0\,. \tag{1.4}$$

Now we insert (1.4) into (1.3) and use (1.1) to obtain from the last sum in (1.3)

$$\begin{aligned}
&-\sum_{i=1}^{2} \left[\rho_i \left(\psi_i\right)_t + \rho_i u^{(i)} \cdot \nabla \psi_i\right] \\
&= -\sum_{i=1}^{2} \left[\rho_i \Big(c_i \Psi(c_1\rho_1 + c_2\rho_2)\Big)_t + \rho_i u^{(i)} \cdot \nabla \Big(c_i \Psi(c_1\rho_1 + c_2\rho_2)\Big)\right] \\
&= -\sum_{i=1}^{2} \Big[c_i\rho_i \Psi'(c_1\rho_1 + c_2\rho_2)\big(- c_1 \operatorname{div}\left(\rho_1 u^{(1)}\right) - c_2 \operatorname{div}\left(\rho_2 u^{(2)}\right) \big) \\
&\qquad + c_i\rho_i u^{(i)} \cdot \left(\nabla(c_1\rho_1) + \nabla(c_2\rho_2)\right) \Psi'(c_1\rho_1 + c_2\rho_2)\Big] \\
&= \sum_{i=1}^{2} c_i\rho_i (c_1\rho_1 + c_2\rho_2)\Psi'(c_1\rho_1 + c_2\rho_2) \operatorname{div} u^{(i)} \\
&\quad - \Psi'(c_1\rho_1 + c_2\rho_2)\Big(c_2\rho_2 \nabla(c_1\rho_1) - c_1\rho_1 \nabla(c_2\rho_2)\Big) \cdot \left(u^{(2)} - u^{(1)}\right)\,.
\end{aligned}$$

Setting

$$P_i(\rho) = c_i\rho_i (c_1\rho_1 + c_2\rho_2)\Psi'(c_1\rho_1 + c_2\rho_2)\,, \tag{1.5}$$

P_i denoting the pressure for the ith species, we obtain finally from (1.3)

$$\begin{aligned}
\sum_{i=1}^{2} \left[T^{(i)} + P_i(\rho) Id\right] : \nabla u^{(i)} + \Big[J^{(1)} - \Psi'(c_1\rho_1 + c_2\rho_2)\big(c_2\rho_2 \nabla(c_1\rho_1) - c_1\rho_1 \nabla(c_2\rho_2)\big)\Big] \cdot \\
\cdot \left(u^{(2)} - u^{(1)}\right) \geq 0\,,
\end{aligned} \tag{1.6}$$

where Id denotes the identity tensor. We set

$$\begin{aligned}
\sigma^{(i)} &:= T^{(i)} + P_i(\rho) Id\,, \\
G &:= J^{(1)} - \Psi'(c_1\rho_1 + c_2\rho_2)\left(c_2\rho_2 \nabla(c_1\rho_1) - c_1\rho_1 \nabla(c_2\rho_2)\right)\,.
\end{aligned}$$

Furthermore, we set

$$\sigma^{(i)} = 2\mu_{i1} D\left(u^{(1)}\right) + 2\mu_{i2} D\left(u^{(2)}\right) + \lambda_{i1}\operatorname{div} u^{(1)} Id + \lambda_{i2} \operatorname{div} u^{(2)} Id\,, \tag{1.7}$$

$$G = a\left(\rho_1,\rho_2,\left|u^{(1)}-u^{(2)}\right|\right)\left(u^{(2)}-u^{(1)}\right) \tag{1.8}$$

with constant viscosity coefficients μ_{ik} and λ_{ik}. D denotes the symmetric part of the gradient, $Dw = \frac{1}{2}\left(\nabla w + (\nabla w)^T\right)$ for $w\colon \mathbb{R}^3 \to \mathbb{R}^3$. We require that for a certain $c_0 > 0$

$$\sum_{i=1}^{2} \sigma^{(i)} : \nabla u^{(i)} \geq c_0 |\nabla u|^2 \,.\text{[1]} \tag{1.9}$$

We assume further that

$$a\left(\rho_1,\rho_2,\left|u^{(1)}-u^{(2)}\right|\right) \geq 0\,. \tag{1.10}$$

Then the inequality (1.6) is automatically fulfilled.

This means that the system (1.1)–(1.2) with

$$T^{(i)} = -P_i(\rho) Id + \sigma^{(i)}\,, \tag{1.11}$$

$$\begin{aligned} J^{(1)} &= a\left(\rho_1,\rho_2,\left|u^{(1)}-u^{(2)}\right|\right)\left(u^{(2)}-u^{(1)}\right) \\ &\quad + \Psi'(c_1\rho_1 + c_2\rho_2)\left(c_2\rho_2 \nabla(c_1\rho_1) - c_1\rho_1\nabla(c_2\rho_2)\right) \end{aligned} \tag{1.12}$$

is thermo-mechanically consistent, in other words that the system fulfills the basic energy estimates.

Indeed, if we consider the system (1.1)–(1.2) e.g. in a bounded domain $\Omega \subset \mathbb{R}^3$ imposing appropriate initial and boundary conditions and test (1.2) for $i = 1$ by $u^{(1)}$ and (1.2) for $i = 2$ by $u^{(2)}$, using integration by parts and the continuity equations (1.1) for $i = 1$ and $i = 2$, and sum over i, we obtain (at least formally) the following inequality

$$\begin{aligned} &\frac{d}{dt}\int_\Omega (c_1\rho_1 + c_2\rho_2)\Psi(c_1\rho_1 + c_2\rho_2)\,dx + \sum_{i=1}^{2}\frac{d}{dt}\left(\frac{1}{2}\int_\Omega \rho_i |u^{(i)}|^2\,dx\right) \\ &+ c_0\int_\Omega |\nabla u|^2\,dx + \int_\Omega a(\rho_1,\rho_2,|u^{(1)}-u^{(2)}|)|u^{(1)}-u^{(2)}|^2\,dx \\ &\qquad \leq \sum_{i=1}^{2}\int_\Omega \rho_i u^{(i)}\cdot f^{(i)}\,dx\,. \end{aligned} \tag{1.13}$$

[1]This is equivalent in terms of the viscosities to

$$\begin{aligned} \mu_{11} > 0, \mu_{22} > 0, 2\mu_{11} + \lambda_{11} > 0, 2\mu_{22} + \lambda_{22} &> 0\,, \\ 4\mu_{11}\mu_{22} - (\mu_{12} + \mu_{21})^2 &> 0\,, \\ 4\left(2\mu_{11} + \lambda_{11}\right)\left(2\mu_{22} + \lambda_{22}\right) - \left(2\mu_{12} + \lambda_{12} + 2\mu_{21} + \lambda_{21}\right)^2 &> 0\,. \end{aligned}$$

In this thesis, however, we consider an approximation of the above model by neglecting the second part in (1.12). From the mathematical point of view we cannot expect to obtain sufficient information in order to pass to the limit in the term containing $c_2\rho_2\nabla(c_1\rho_1) - c_1\rho_1\nabla(c_2\rho_2)$ because the densities fulfill only a first-order equation. But it is also reasonable to neglect the term from the point of view of physics. First of all, it is very difficult to identify the term experimentally. Moreover, numerical simulations of flows in special geometries have shown that there is no significant difference in the result if the second term in (1.12) is considered or not (cf. also [Raj00]).

Thus, we consider instead of $J^{(i)}$ given by (1.12) only interaction terms of the form

$$I^{(i)} = (-1)^{i+1} a\left(\rho_1, \rho_2, \left|u^{(1)} - u^{(2)}\right|\right)\left(u^{(2)} - u^{(1)}\right) . \tag{1.14}$$

Unfortunately, with this choice of the interaction terms there is in general no energy estimate available for the system (1.1)–(1.2).

By introducing the notation

$$L_{ik} = -\mu_{ik}\Delta - (\lambda_{ik} + \mu_{ik})\nabla \operatorname{div} ,$$

the balance equations write as follows for $i = 1, 2$:

$$(\rho_i)_t + \operatorname{div}\left(\rho_i u^{(i)}\right) = 0 , \tag{1.15}$$

$$(\rho_i u^{(i)})_t + \sum_{k=1}^{2} L_{ik} u^{(k)} + \operatorname{div}\left(\rho_i u^{(i)} \otimes u^{(i)}\right) = -\nabla P_i(\rho) + \rho_i f^{(i)} + I^{(i)} , \tag{1.16}$$

complemented by suitable initial and boundary conditions, where the operators L_{ik} are assumed to fulfill the ellipticity condition (1.9).

In this thesis, we consider interaction terms $I^{(i)}$ of the form (1.14), where $a > 0$ is in general chosen to be constant, or a may sometimes depend – in a possibly nonlinear way – on ρ.

The pressure is given by (1.5). Different choices of the function Ψ lead to different pressure laws. For instance the choice $\Psi(c_1\rho_1 + c_2\rho_2) = \log(c_1\rho_1 + c_2\rho_2)$ leads to the pressure law

$$P_i(\rho) = c_i\rho_i , \quad c_i > 0 .$$

We will deal with this case in Chapter 2.

Let us remark that in this case we have similar estimates for the approximated model as for the full model. The only difference is that we have instead of the term

$$\frac{d}{dt}\int_\Omega (c_1\rho_1 + c_2\rho_2)\log(c_1\rho_1 + c_2\rho_2)\, dx$$

in inequality (1.13) the sum

$$\sum_{i=1}^{2} \frac{d}{dt} c_i \int_\Omega \rho_i \log \rho_i \, dx \, .$$

For stationary flows we obtain in this case the same inequality for both models.

In general, however, the pressure $P_i(\rho)$ depends on *both* ρ_1 *and* ρ_2. We consider in Chapter 3, 4 and 5 Ψ of the form $\Psi(c_1\rho_1 + c_2\rho_2) = (c_1\rho_1 + c_2\rho_2)^{\gamma-1}$, which leads to the pressure law

$$P_i(\rho) = \tilde{c}_i \rho_i \left(c_1\rho_1 + c_2\rho_2\right)^{\gamma-1} \, , \quad \tilde{c}_i > 0 \, .$$

There is a broad interest from engineers in studying and understanding mixture models of type (1.15)–(1.16) since some numerical tests have shown good agreement of the models with real experiments – e.g. for lubrication with emulsions (cf. [ASCRS93], [WASRS93], [WSR93], [CASRS93], [RT95]). Nevertheless, there is up to now almost no mathematical theory in more than one space dimension due to mathematical difficulties. (Mixtures in one space dimension were treated e.g. in [Zlo95] and [KP78].)

First results for the mixture model in more than one space dimension were achieved by Frehse, Goj, Málek. In [FGM02] and [FGM04a] the existence of weak solutions to a Stokes-like model for mixtures in the whole space $\mathbb{R}^3$ is shown. The paper [FGM04b] treats the uniqueness of solutions to this model if the forces and interaction terms are zero. Concerning the Stokes-like system, there are up to now no results available for bounded domains.

Frehse and Weigant deal with the quasi-stationary case in a bounded domain Ω with special boundary conditions (cf. [FW04]). In this case one has even Lipschitz-continuous densities.

But there is still no existence result available for the general case of the mixture model (1.15)–(1.16) with consideration of the convective terms.

Alternative models for mixtures use only *one* density function and *one* velocity field for the whole mixture. In contrast to the model (1.15)–(1.16), the effects of mutual interactions of the individual components cannot be captured by these models. With respect to mathematical theory, there is a lot more known about these kinds of models (cf. [NP95], [NPD97], [Des97], [Lio96]).

It is not astonishing that the mathematical theory for the model for mixtures of compressible fluids (1.15)–(1.16) is not very developed since for the classical case of the Navier–Stokes equations for *one* compressible fluid real progress was achieved only in the last ten years. For the Navier–Stokes system in the isentropic case

$$\rho_t + \operatorname{div}(\rho u) = 0 \, , \tag{1.17}$$

$$(\rho u)_t - \mu\Delta u - (\lambda + \mu)\nabla \operatorname{div} u + \operatorname{div}(\rho u \otimes u) = -a\nabla\rho^\gamma + \rho f \tag{1.18}$$

for the density $\rho\colon [0,T]\times\Omega\to\mathbb{R}$ and the velocity field $u\colon [0,T]\times\Omega\to\mathbb{R}^3$, $\Omega\subset\mathbb{R}^3$, global existence of weak solutions for large data was first shown by P.-L. Lions in his book [Lio98], after announcements of the results in [Lio93a] and [Lio93b]. He proved the existence of weak solutions in three dimensions under the constraint that $\gamma\geq\frac{9}{5}$, in which case the density belongs to the space L^2. This result was improved by E. Feireisl et al. in the papers [Fei01], [FNP01], where the authors succeeded to obtain the existence of weak solutions in the evolutionary case for $\gamma>\frac{3}{2}$. Now, E. Feireisl proved even some results concerning the Navier–Stokes–Fourier system where also temperature dependence is taken into account (cf. [Fei04]).

In the steady case, the existence of weak solutions to the three-dimensional system is only known for

$$\gamma>\frac{3}{2}\ \text{if}\ \operatorname{curl} f=0\,,\ \text{and}\ \gamma>\frac{5}{3}\ \text{if}\ \operatorname{curl} f\neq 0\,,$$

see e.g. the articles by Novotný et al. ([Nov96], [Nov98], [NN02]) or the monograph by Novotný and Straškraba [NS04].

In this thesis we will make use of and adapt some of the techniques from the theory of compressible flow as for instance the equation for the effective viscous flux, which plays a key role in proving the compactness of approximate densities.

In addition to the difficulties occuring in the case of the Navier–Stokes equations for compressible flow (in particular due to the nonlinearity of the pressure), we have to deal with even more complexities in the case of mixtures due to aspects like the coupling of the shear forces or the pressure law. In contrast to the one-component case, there is in general no energy estimate available for the mixture model (1.15)–(1.16), as mentioned above.

We want to illustrate this by comparing the system (1.15)–(1.16) to the Navier–Stokes equations for isentropic compressible flow. In the one-component case, we obtain an energy estimate by formally testing equation (1.18) (complemented with suitable initial and boundary conditions) by u and using the continuity equation (1.17) for the pressure term:

$$\begin{aligned}
-a\int_\Omega \nabla\rho^\gamma\cdot u\,dx &= -a\gamma\int_\Omega \rho^{\gamma-1}\nabla\rho\cdot u\,dx\\
&= -\frac{a\gamma}{\gamma-1}\int_\Omega \rho u\cdot\nabla\rho^{\gamma-1}\,dx\\
&= \frac{a\gamma}{\gamma-1}\int_\Omega \operatorname{div}(\rho u)\rho^{\gamma-1}\,dx\\
&= -\frac{a}{\gamma-1}\int_\Omega (\rho^\gamma)_t\,dx\,.
\end{aligned}$$

We obtain

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega \rho|u|^2\,dx + \frac{a}{\gamma-1}\int_\Omega \rho^\gamma\,dx\right)$$
$$+\mu\int_\Omega |\nabla u|^2\,dx + (\lambda+\mu)\int_\Omega (\operatorname{div} u)^2\,dx \le \int_\Omega \rho u\cdot f\,dx\,.$$

The lower-order term $\int_\Omega \rho u\cdot f\,dx$ can be treated by Hölder's inequality. If we impose for instance $f\in L^1(0,T;L^{\frac{2\gamma}{\gamma-1}}(\Omega;\mathbb{R}^3))$, we obtain

$$\int_\Omega \rho u\cdot f\,dx \le \|\rho u\|_{L^{\frac{2\gamma}{\gamma+1}}}\|f\|_{L^{\frac{2\gamma}{\gamma-1}}} \le \|\sqrt{\rho}\|_{L^{2\gamma}}\|\sqrt{\rho}u\|_{L^2}\|f\|_{L^{\frac{2\gamma}{\gamma-1}}}\,.$$

When dealing with flows of mixtures described by (1.15)–(1.16) and considering e.g. the typical pressure law

$$P_i(\rho) = \tilde{c}_i\rho_i\,(c_1\rho_1 + c_2\rho_2)^{\gamma-1}\ ,$$

we cannot expect that the term

$$\tilde{c}_i\int \rho_i\,(c_1\rho_1+c_2\rho_2)^{\gamma-1}\operatorname{div} u^{(i)}\,dx$$

can be treated with the aid of the continuity equation (1.15) because the pressure depends on *both* density functions ρ_1 *and* ρ_2 and we do not have the second part of the interaction term $J^{(i)}$ at our disposal.

Therefore, in [FGM04a] the authors develop a different method for obtaining estimates in the case of the Stokes problem, which uses the effective viscous flux. But to current knowledge it seems that this method works only in the whole space $\mathbb{R}^3$.

Thus, we investigate in this thesis (in Chapter 2) a Stokes-like model where we can get estimates directly from the equations such that we can treat also the case of a bounded domain with standard no-slip and slip boundary conditions.

In this thesis we will deal with the following problems:

In Chapter 2, we treat a Stokes-like model with a linear pressure law in a bounded domain, i.e. we consider a steady model where the quantities are independent of time such that $(\rho_i)_t = 0$ and $(\rho_i u^{(i)})_t = 0$ and neglect also the convective terms, $\operatorname{div}(\rho_i u^{(i)}\otimes u^{(i)}) = 0$. The Stokes-like problem is a good approximation of the full system for strongly viscous fluids and in the case of small accelerations. We prove the existence of weak solutions to this model, slight regularity properties of the solutions and the strong convergence of the approximate densities. The results presented here are the first ones for the Stokes-like system for mixtures in a bounded domain.

Moreover, they are the first results for the model for mixtures in a bounded domain with standard no-slip or slip boundary conditions.

In the third chapter, we deal with the steady mixture model taking into account also the convective terms. We prove the compactness of solutions to this model under the assumption of suitable estimates. More precisely, we consider a sequence of solutions to the equations fulfilling certain estimates and prove that the limit of this sequence is a solution as well.
The compactness – or weak sequential stability – is considered to be the main step in an existence proof. However, for our model it is not obvious how to obtain approximate solutions which satisfy appropriate estimates.

In the last two chapters, we present new methods for proving estimates. The findings presented here are regularity results, but not existence results.

The fourth chapter deals with L^p-estimates for the mixture model with consideration of the convective terms in the steady case. These estimates were developed in [FGS04] for the case of the steady Navier–Stokes equations for compressible isentropic flow and are adapted in this thesis to treat the equations describing mixtures of compressible fluids.
In the chapter we consider a pressure law which behaves like $|\rho|^\gamma$ with $\frac{5}{4} < \gamma \leq 5$. Under the assumption that we have solutions of the momentum equation fulfilling suitable estimates, we prove that

$$\rho_i |u^{(i)}|^2 \in L_{loc}^{\frac{6\gamma}{5+2\gamma}}(\Omega) \text{ and } \rho_i \in L_{loc}^{\frac{6\gamma^2}{5+2\gamma}}(\Omega)\,.$$

This kind of estimates is important for applying the technique introduced by Feireisl ([Fei01]) for proving the compactness of the densities. In the one-component case, these findings are an improvement of known estimates for $\frac{5}{4} < \gamma \leq \frac{5}{3}$.

In the fifth chapter, we present a new exponential estimate for the densities in the case of the Stokes problem with a pressure law $P(\rho) \sim |\rho|^\gamma, \gamma \geq 2$. It is still open to prove an L^∞-estimate for the densities in the case of the Stokes-like system. The estimates presented in Chapter 5 have to be seen as a step in this direction.

Notation

We use standard notation throughout this thesis.

For $a, b \in \mathbb{R}^3$ we denote the inner product by $a \cdot b = \sum_{i=1}^3 a_i b_i$, $|a| = (a \cdot a)^{1/2}$ denotes the norm in $\mathbb{R}^3$, and analogously in $\mathbb{R}^n$ for different dimensions. The tensor product of a and b is written by

$$a \otimes b = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix} .$$

Moreover, for 3×3-matrices $a, b \in \mathbb{R}^{3\times 3}$ we write $a : b = \sum_{i,j=1}^{3} a_{ij}b_{ij}$.

The open ball with radius R and center $x_0 \in \mathbb{R}^3$ is denoted by

$$B_R(x_0) = \{x \in \mathbb{R}^3 | |x - x_0| < R\} ,$$

especially for $x_0 = 0$ we write $B_R := B_R(0)$.

For a function $v\colon [0, T] \times \Omega \to \mathbb{R}$ we use the following standard notation for the partial derivatives

$$\partial_k v := \frac{\partial v}{\partial x_k} , \quad k = 1, 2, 3 ,$$

$$v_t := \partial_t v := \frac{\partial v}{\partial t} .$$

∇v denotes the vector of the first derivatives with respect to x,

$$\Delta v = \sum_{j=1}^{3} \partial_{jj}^2 v \text{ is the Laplace operator.}$$

For vector-valued functions $w\colon [0, T] \times \Omega \to \mathbb{R}^3$, $w = (w_1, w_2, w_3)^T$, we understand the symbols ∂_t, Δ etc. component-wise.

The divergence and the curl of w are denoted by

$$\operatorname{div} w = \sum_{j=1}^{3} \partial_j w_j \quad \text{and} \quad \operatorname{curl} w = \begin{pmatrix} \partial_2 w_3 - \partial_3 w_2 \\ \partial_3 w_1 - \partial_1 w_3 \\ \partial_1 w_2 - \partial_2 w_1 \end{pmatrix} .$$

In this thesis, we deal with the usual Lebesgue spaces $L^p(\Omega), 1 \le p \le \infty$, of equivalence classes of functions $v\colon \Omega \to \mathbb{R}$ which are measurable and for which $|v|^p$ is integrable, with the norm

$$\|v\|_{L^p} = \left(\int_\Omega |v|^p \, dx \right)^{\frac{1}{p}} , \qquad 1 \le p < \infty ,$$

$$\|v\|_{L^\infty} = \operatorname*{ess\,sup}_{x\in\Omega} |v(x)| .$$

$L^p(\Omega; \mathbb{R}^3)$ is the space of functions $w\colon \Omega \to \mathbb{R}^3$ with components w_1, w_2, w_3 belonging to $L^p(\Omega)$, and analogously $L^p(\Omega; \mathbb{R}^{3\times 3})$.
Sometimes we just write L^p instead of $L^p(\Omega)$ or $L^p(\Omega; \mathbb{R}^3)$ etc.

The Sobolev space $W^{k,p}(\Omega), 1 \leq p < \infty, k \in \mathbb{N}$, consists of all functions $v \in L^p(\Omega)$ which possess partial derivatives D^α in the weak sense up to order k belonging in Ω to the Lebesgue class L^p. The norm is given by

$$\|v\|_{W^{k,p}} = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha v|^p \, dx \right)^{\frac{1}{p}} .$$

The space $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the $W^{k,p}$-norm, where $C_0^\infty(\Omega)$ denotes the space of smooth functions with compact support in Ω.

For $p = 2$ we will make use of the abbreviation $H^k := W^{k,2}$ and $H_0^k := W_0^{k,2}$.

For vector-valued functions we use the notation $W^{k,p}(\Omega; \mathbb{R}^3)$ and $H^k(\Omega; \mathbb{R}^3)$.

Very often we make use of a generic constant K, which attains, in general, different values at different places.

For reasons of lucidity, if we extract a subsequence from a sequence, we denote the subsequence as the original sequence in order to avoid double subscripts.

Concerning the equations, we will use the following notations for the external force densities, the interaction terms and the pressure:

$$f = \begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix}, I = \begin{pmatrix} I^{(1)} \\ I^{(2)} \end{pmatrix}, P(\rho) = \begin{pmatrix} P_1(\rho) \\ P_2(\rho) \end{pmatrix} \text{ etc.}$$

Acknowlegdements: *First and foremost I would like to thank my advisor Professor Frehse for giving me the opportunity to work in the very interesting field of mixtures of fluids and also for his advice during the completion of this thesis.*

Moreover, I am grateful to Professor Josef Málek, Charles University Prague, for his hospitality and for many helpful discussions concerning the model.

This work was supported by the Research Training Network "Hyperbolic and Kinetic Equations (HYKE)" financed by the European Union in the 5th Framework Program "Improving the Human Potential" (Contract Number: HPRN-CT-2002-00282), which made my stay at the Charles University in Prague (02/2003–06/2003) possible.

Furthermore, financial support by the SFB 611 "Singular Phenomena and Scaling in Mathematical Models" of the University of Bonn is gratefully acknowlegded.

Last but not least, I am much obliged to the German National Academic Foundation (Studienstiftung des deutschen Volkes) for their non-financial support during the last years.

Chapter 2

On a Stokes-like system for mixtures

In this chapter we consider a simplification of the steady mixture model in which we neglect the convective terms in equation (1.16). In analogy to the one-component case for incompressible fluids we call this model a *Stokes-like system for mixtures.*

From the point of view of physics, the Stokes problem is a good approximation for strongly viscous fluids or in the case of slow flows, where one can achieve by a proper scaling a non-dimensional form that allows to neglect the convective term.

In our case, the motivation has to be seen indeed more mathematically. Since there is no mathematical theory available for the mixture model in more than one space dimension, it is, of course, proximate to start like in the classical case of incompressible flow for single continuum with a Stokes-like problem when the full system seems far beyond the scope of known mathematical methods.

In this chapter we investigate the Stokes-like model for mixtures in a bounded open connected domain $\Omega \subset \mathbb{R}^3$. We would like to emphasize that this thesis presents the first results for the Stokes-like system for mixtures in a *bounded* domain.

We consider the following set of equations for the velocity fields $u^{(i)} \colon \Omega \to \mathbb{R}^3$ and the density functions $\rho_i \colon \Omega \to \mathbb{R}$, $\rho_i \geq 0$, $i = 1, 2$:

$$\operatorname{div}\left(\rho_i u^{(i)}\right) = 0 \qquad \text{in } \Omega\,, \tag{2.1}$$

$$\sum_{k=1}^{2} L_{ik} u^{(k)} = -\nabla P_i(\rho) + \rho_i f^{(i)} + I^{(i)} \qquad \text{in } \Omega \tag{2.2}$$

with

$$L_{ik} = -\mu_{ik}\Delta - (\lambda_{ik} + \mu_{ik})\nabla \operatorname{div}\,.$$

Furthermore,

$$\int_\Omega \rho_i \, dx = M > 0 \, , M \text{ given, say } M = 1 \, . \tag{2.3}$$

The operators L_{ik} are assumed to fulfill the following ellipticity condition:

$$\sum_{i,k=1}^{2} \int_\Omega L_{ik} u^{(k)} \cdot u^{(i)} \, dx \geq c_0 \int_\Omega |\nabla u|^2 \, dx \, . \tag{2.4}$$

The equations (2.1)–(2.3) are complemented with no-slip boundary conditions for the velocities:

$$u^{(i)} = 0 \text{ on } \partial\Omega \, . \tag{2.5}$$

Remark: *It is also possible to treat the case of slip boundary conditions for the velocities; the results from this chapter can be proved analogously for this type of boundary conditions as well: For* $i = 1, 2$

$$u^{(i)} \cdot \vec{n} = 0 \text{ on } \partial\Omega \tag{2.6}$$

plus natural boundary conditions. Here, $\vec{n}$ *denotes the outer normal vector. The natural boundary conditions arising from the use of test functions* φ *with* $\varphi \cdot \vec{n}\big|_{\partial\Omega} = 0$ *in the weak formulation of the momentum equation (2.2) are given by*

$$T^{(i)} \vec{n} \cdot \vec{t^s}\big|_{\partial\Omega} = 0 \text{ for } s = 1, 2 \, , \tag{2.7}$$

where $(\vec{t^1}, \vec{t^2})$ *is a basis of the tangent space.*
Since the Cauchy stress tensor in our case has the form

$$T^{(i)} = -P_i(\rho) Id + 2\mu_{i1} D(u^{(1)}) + 2\mu_{i2} D(u^{(2)}) + \lambda_{i1} \operatorname{div} u^{(1)} Id + \lambda_{i2} \operatorname{div} u^{(2)} Id \, ,$$

we obtain from condition (2.7) using $Id \, \vec{n} \cdot \vec{t^s} = 0$

$$\left(2\mu_{i1} D(u^{(1)}) + 2\mu_{i2} D(u^{(2)})\right) \vec{n} \cdot \vec{t^s}\big|_{\partial\Omega} = 0 \text{ for } s = 1, 2 \, .$$

In components, this is written as

$$\sum_{j,k=1}^{3} \vec{t^s_j} \left\{ \mu_{i1} \left(\partial_j u_k^{(1)} + \partial_k u_j^{(1)} \right) + \mu_{i2} \left(\partial_j u_k^{(2)} + \partial_k u_j^{(2)} \right) \right\} \vec{n}_k \big|_{\partial\Omega} = 0 \tag{2.8}$$

for $s = 1, 2$, $(\vec{t^1}, \vec{t^2})$ *being a basis of the tangent space.*
Thus, we have with (2.6) and (2.8) also in the case of slip boundary conditions for each of the velocity fields $u^{(i)}, i = 1, 2$, *three conditions which are prescribed at the boundary.*

The pressure is assumed to be linear:

$$P_i(\rho) = c_i \rho_i \tag{2.9}$$

with constants $c_i > 0, i = 1, 2$. (As mentioned in the introduction, this pressure law is obtained by choosing a logarithmic Helmholtz potential Ψ.)

The interaction terms are given by

$$I^{(1)} = -I^{(2)} = a(\rho_1, \rho_2, |u^{(1)} - u^{(2)}|)\,(u^{(2)} - u^{(1)})\,,$$

where we assume the factor $a > 0$ to be constant, i.e.

$$I^{(i)} = (-1)^{i+1} a \left(u^{(2)} - u^{(1)}\right)\,. \tag{2.10}$$

Remark: *It is also possible to treat interaction terms of the form*

$$I^{(i)} = (-1)^{i+1} a(\rho) \left(u^{(2)} - u^{(1)}\right) \tag{2.11}$$

with the factor a depending in a possibly nonlinear way on the densities ρ_i, which makes sense from the point of view of physics. We have to underline that the proof in the first section of this chapter is not sufficient to show the existence of weak solutions for the model with this kind of interaction term, but together with the third section, where the compactness of approximate densities is shown, we obtain the existence for this case as well.

The Stokes-like system was already treated in the papers [FGM02], [FGM04a] and [FGM04b] as existence of weak solutions and uniqueness in the case of zero forces is concerned. In these articles the problem in the whole space $\mathbb{R}^3$ is considered with the following conditions at infinity:

$$u^{(i)} \to 0\,, \quad \rho_i \to \rho_{i\infty} > 0 \text{ as } |x| \to \infty \text{ for } i = 1, 2\,.$$

The pressure considered there is basically given by

$$P_i(\rho) = c_i \rho_i \left(\frac{\rho_1}{\rho_{1,ref}} + \frac{\rho_2}{\rho_{2,ref}} \right)^{\gamma - 1}$$

with $\gamma > 1, c_i > 0, i = 1, 2$, and positive reference densities $\rho_{1,ref}, \rho_{2,ref}$. (More precisely, the pressure has to fulfill a monotonicity, a coerciveness and a growth condition.)

In this case, the authors were able to obtain estimates for the densities in $L^2(\mathbb{R}^3) \cap L^{2\gamma}(\mathbb{R}^3)$ and for the velocity fields in $H_0^1(\mathbb{R}^3; \mathbb{R}^3)$ and to prove the existence of weak solutions. The way of estimating the densities in these articles (using the equation for the effective viscous flux) does up to now not work in the case of a bounded domain $\Omega \subset \mathbb{R}^3$ with standard no-slip or slip boundary conditions for the velocities.

Thus, in the case considered here we are fortunate that we can obtain estimates for $u^{(i)}$ and ρ_i in suitable spaces in a more direct way from the equations.

Moreover, we want to remark that in [FGM02] and [FGM04b] the authors were only able to deal with interaction terms with *sublinear* growth in $(u^{(2)} - u^{(1)})$, whereas we are coping here with interaction terms which are linear in $(u^{(2)} - u^{(1)})$.

In this chapter, we prove the following results for the Stokes-like system in a bounded domain $\Omega \subset \mathbb{R}^3$ with a linear pressure law:

- existence of weak solutions,
- estimates for ρ_i and $\nabla u^{(i)}$ in L^p for all $1 \leq p < \infty$,
- compactness of the densities, i.e. strong convergence of approximate densities to a solution of the equations under consideration.

2.1 Existence of weak solutions

In this section, we prove the existence of weak solutions to the system (2.1)–(2.3), (2.5).

By a *weak solution of the system (2.1)–(2.3), (2.5)* we mean a pair $(\rho, u), \rho = (\rho_1, \rho_2)^T, u = (u^{(1),T}, u^{(2),T})^T$, such that $\rho_i \in L^2(\Omega), u^{(i)} \in H_0^1(\Omega; \mathbb{R}^3), \rho_i \geq 0, i = 1, 2,$ fulfilling for $i = 1, 2$

$$\int_\Omega \rho_i \, dx = 1 \,,$$

and

$$\text{for all } \zeta \in H^1(\Omega) : \int_\Omega \rho_i u^{(i)} \cdot \nabla \zeta \, dx = 0 \,,$$

and for all functions $\varphi \in H_0^1(\Omega; \mathbb{R}^3)$:

$$\sum_{k=1}^{2} \left(\mu_{ik} \int_\Omega \nabla u^{(k)} : \nabla \varphi \, dx + (\lambda_{ik} + \mu_{ik}) \int_\Omega \operatorname{div} u^{(k)} \operatorname{div} \varphi \, dx \right) = c_i \int_\Omega \rho_i \operatorname{div} \varphi \, dx$$
$$+ \int_\Omega \rho_i f^{(i)} \cdot \varphi \, dx + \int_\Omega I^{(i)} \cdot \varphi \, dx \,.$$

Remark: *In the case of boundary conditions of type $u^{(i)} \cdot \vec{n}\big|_{\partial\Omega} = 0$ complemented with natural boundary conditions, the test functions for the momentum equation have to be chosen as $\varphi \in H^1(\Omega; \mathbb{R}^3) \cap \big(\varphi \cdot \vec{n}\big|_{\partial\Omega} = 0\big)$.*

The following theorem contains the main assertion which we will prove in this section.

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain. Let $f^{(i)} \in L^\infty(\Omega;\mathbb{R}^3), i = 1,2$. There exists a weak solution (ρ, u), $\rho = (\rho_1, \rho_2)^T, u = (u^{(1),T}, u^{(2),T})^T$, of the system (2.1)–(2.3), (2.5) with L_{ik} satisfying the ellipticity condition (2.4), the pressure P being of the form (2.9) and the interaction terms of the form (2.10).*
The solution fulfills

$$\rho_i \in L^2(\Omega), u^{(i)} \in H_0^1(\Omega;\mathbb{R}^3), \rho_i \geq 0 \text{ for } i = 1,2. \tag{2.12}$$

In order to prove this theorem, we construct solutions of the following system of approximative equations in Ω

$$-\sigma\Delta\rho_i^{\alpha_0,\alpha,\sigma} + \operatorname{div}\left(\rho_i^{\alpha_0,\alpha,\sigma} u^{(i)}_{\alpha_0,\alpha,\sigma}\right) + \alpha\rho_i^{\alpha_0,\alpha,\sigma} + \alpha_0|\rho_i^{\alpha_0,\alpha,\sigma}|^{s-1}\rho_i^{\alpha_0,\alpha,\sigma} = \frac{\alpha}{|\Omega|}\,, \tag{2.13}$$

$$\sum_{k=1}^{2} L_{ik} u^{(k)}_{\alpha_0,\alpha,\sigma} = -c_i \nabla\rho_i^{\alpha_0,\alpha,\sigma} + \rho_i^{\alpha_0,\alpha,\sigma} f^{(i)} + I^{(i)}_{\alpha_0,\alpha,\sigma}\,, \tag{2.14}$$

$$u^{(i)}_{\alpha_0,\alpha,\sigma}\big|_{\partial\Omega} = 0\,, \qquad \nabla\rho_i^{\alpha_0,\alpha,\sigma}\cdot\vec{n}\big|_{\partial\Omega} = 0\,, \tag{2.15}$$

for $\alpha_0, \alpha, \sigma > 0$ and s large (at least $s > 3$) and perform the limit process first for $\alpha_0 \to 0$ and then for $\alpha, \sigma \to 0$. Here, $I^{(i)}_{\alpha_0,\alpha,\sigma} = (-1)^{i+1} a\left(u^{(2)}_{\alpha_0,\alpha,\sigma} - u^{(1)}_{\alpha_0,\alpha,\sigma}\right)$ and $|\Omega|$ denotes the measure of the domain Ω.

The viscosity approximation of the continuity equation is a standard approach in the theory of compressible flow (compare e.g. [FNP01], [NS04]). Adding the parabolic term $-\sigma\Delta\rho_i^{\alpha_0,\alpha,\sigma}$ in equation (2.1) assures that one can deal with $\nabla\rho_i^{\alpha_0,\alpha,\sigma}$ for fixed $\sigma > 0$. The additional terms with higher powers of $\rho_i^{\alpha_0,\alpha,\sigma}$ ensure the existence of solutions on the approximate level because they guarantee coerciveness of the equations. The term $\alpha\rho_i^{\alpha_0,\alpha,\sigma}$ together with $\frac{\alpha}{|\Omega|}$ ensures that $\int_\Omega \rho_i \, dx = 1$ holds in the limit.

Remark: *Instead of (2.13) we could use the equation*

$$-\sigma\Delta\rho_i^{\alpha_0,\alpha,\sigma} + \operatorname{div}\left((\rho_i^{\alpha_0,\alpha,\sigma})^+ u^{(i)}_{\alpha_0,\alpha,\sigma}\right) + \alpha\rho_i^{\alpha_0,\alpha,\sigma} + \alpha_0|\rho_i^{\alpha_0,\alpha,\sigma}|^{s-1}\rho_i^{\alpha_0,\alpha,\sigma} = \frac{\alpha}{|\Omega|}\,, \tag{2.16}$$

where $v^+ = \max(v,0)$ denotes the positive part of the function v.

After having established the existence of solutions to the system (2.16), (2.14), (2.15), we can test by the negative part $(\rho_i^{\alpha_0,\alpha,\sigma})^- = \min(\rho_i^{\alpha_0,\alpha,\sigma}, 0)$ in (2.16). As a result, one obtains

$$(\rho_i^{\alpha_0,\alpha,\sigma})^- = 0\,,\ \text{i.e. } \rho_i^{\alpha_0,\alpha,\sigma} \geq 0\,.$$

Then we can omit the $(\cdot)^+$ and have solutions to

$$-\sigma\Delta\rho_i^{\alpha_0,\alpha,\sigma} + \operatorname{div}\left(\rho_i^{\alpha_0,\alpha,\sigma} u^{(i)}_{\alpha_0,\alpha,\sigma}\right) + \alpha\rho_i^{\alpha_0,\alpha,\sigma} + \alpha_0|\rho_i^{\alpha_0,\alpha,\sigma}|^{s-1}\rho_i^{\alpha_0,\alpha,\sigma} = \frac{\alpha}{|\Omega|}$$

with $\rho_i^{\alpha_0,\alpha,\sigma} \geq 0$.

We use a different approach to show the nonnegativity of the densities, using a dual problem.

We prove the existence of weak solutions to the approximate system (2.13)–(2.15) and show that the solutions fulfill certain estimates, which allow to pass to the limit in the equations.

Since the equations (2.1)–(2.2) are linear with respect to the density, weak convergence of the approximate densities in L^p is sufficient to pass to the limit in the equations. Thus, it is sufficient to obtain estimates for the approximate densities in L^p spaces.

A *weak solution of the equations (2.13)–(2.15)* is a pair $(\rho^{\alpha_0,\alpha,\sigma}, u_{\alpha_0,\alpha,\sigma})$, $\rho^{\alpha_0,\alpha,\sigma} = (\rho_1^{\alpha_0,\alpha,\sigma}, \rho_2^{\alpha_0,\alpha,\sigma})^T$, $u_{\alpha_0,\alpha,\sigma} = (u_{\alpha_0,\alpha,\sigma}^{(1),T}, u_{\alpha_0,\alpha,\sigma}^{(2),T})^T$, such that $\rho_i^{\alpha_0,\alpha,\sigma} \in L^{s+1}(\Omega) \cap H^1(\Omega)$, $u_{\alpha_0,\alpha,\sigma}^{(i)} \in H_0^1(\Omega;\mathbb{R}^3)$, $\rho_i^{\alpha_0,\alpha,\sigma} \geq 0, i = 1,2$, satisfying (2.15) in the sense of traces and fulfilling for $i = 1,2$:

$$\text{for all functions } \zeta \in H^1(\Omega) \cap L^{s+1}(\Omega):$$

$$\sigma \int_\Omega \nabla \rho_i^{\alpha_0,\alpha,\sigma} \cdot \nabla \zeta \, dx - \int_\Omega \rho_i^{\alpha_0,\alpha,\sigma} u_{\alpha_0,\alpha,\sigma}^{(i)} \cdot \nabla \zeta \, dx + \alpha \int_\Omega \rho_i^{\alpha_0,\alpha,\sigma} \zeta \, dx$$
$$+ \alpha_0 \int_\Omega |\rho_i^{\alpha_0,\alpha,\sigma}|^{s-1} \rho_i^{\alpha_0,\alpha,\sigma} \zeta \, dx = \frac{\alpha}{|\Omega|} \int_\Omega \zeta \, dx \,,$$

$$\text{and for all functions } \varphi \in H_0^1(\Omega;\mathbb{R}^3):$$

$$\sum_{k=1}^{2} \left(\mu_{ik} \int_\Omega \nabla u_{\alpha_0,\alpha,\sigma}^{(k)} : \nabla \varphi \, dx + (\lambda_{ik} + \mu_{ik}) \int_\Omega \operatorname{div} u_{\alpha_0,\alpha,\sigma}^{(k)} \operatorname{div} \varphi \, dx \right)$$
$$= c_i \int_\Omega \rho_i^{\alpha_0,\alpha,\sigma} \operatorname{div} \varphi \, dx + \int_\Omega \rho_i^{\alpha_0,\alpha,\sigma} f^{(i)} \cdot \varphi \, dx + \int_\Omega I_{\alpha_0,\alpha,\sigma}^{(i)} \cdot \varphi \, dx \,.$$

Remark: *For slip boundary conditions* $u^{(i)} \cdot \vec{n}\big|_{\partial\Omega} = 0$ *plus natural boundary conditions we have to choose in the second equation the test space* $H^1(\Omega;\mathbb{R}^3) \cap \big(\varphi \cdot \vec{n}\big|_{\partial\Omega} = 0\big)$.

We want to prove the following

Lemma 2.1 *There exist weak solutions* $(\rho^{\alpha_0,\alpha,\sigma}, u_{\alpha_0,\alpha,\sigma})$ *of (2.13)–(2.15),* $\rho_i^{\alpha_0,\alpha,\sigma} \in H^1(\Omega) \cap L^{s+1}(\Omega)$, $\rho_i^{\alpha_0,\alpha,\sigma} \geq 0$, $u_{\alpha_0,\alpha,\sigma}^{(i)} \in H_0^1(\Omega;\mathbb{R}^3)$, $i = 1,2$.
The solutions fulfill the following estimates, which are independent of α_0, α *and* σ*:*

$$\|\rho^{\alpha_0,\alpha,\sigma}\|_{L^2} \leq K \,, \quad \|u_{\alpha_0,\alpha,\sigma}\|_{H_0^1} \leq K \,. \tag{2.17}$$

Proof:
The proof is organized as follows: In the first step, we prove the existence of weak solutions to the approximative system.
Secondly, we show that the approximate densities are nonnegative.
Finally, in the third step, we derive estimates for the solutions which do not depend on the approximation parameters α_0, α and σ.

For simplicity of notation we write in the following proof ρ and u instead of $\rho^{\alpha_0,\alpha,\sigma}$ and $u_{\alpha_0,\alpha,\sigma}$.

Step 1: Existence of weak solutions (ρ, u)

We test (2.14) by $u^{(i)}$ and obtain for $i = 1, 2$

$$\int_\Omega \sum_{k=1}^{2} L_{ik} u^{(k)} \cdot u^{(i)}\, dx = -c_i \int_\Omega u^{(i)} \cdot \nabla \rho_i \, dx + \int_\Omega \rho_i f^{(i)} \cdot u^{(i)} \, dx + \int_\Omega I^{(i)} \cdot u^{(i)} \, dx\,.$$

The interaction terms give the following contribution:

$$\text{For } i = 1: \qquad \int_\Omega a\left(u^{(2)} - u^{(1)}\right) \cdot u^{(1)}\, dx = -\int_\Omega a\left(u^{(1)} - u^{(2)}\right) \cdot u^{(1)}\, dx\,,$$

$$\text{for } i = 2: \qquad \int_\Omega a\left(u^{(1)} - u^{(2)}\right) \cdot u^{(2)}\, dx\,.$$

The sum over i equals

$$\sum_{i=1}^{2} \int_\Omega I^{(i)} \cdot u^{(i)}\, dx = -\int_\Omega a\left|u^{(1)} - u^{(2)}\right|^2\, dx\,.$$

The integral coming from the pressure term gives

$$\begin{aligned} \left|-c_i \int_\Omega u^{(i)} \cdot \nabla \rho_i \, dx\right| &= \left|c_i \int_\Omega \operatorname{div} u^{(i)} \rho_i \, dx\right| \\ &\le \varepsilon \int_\Omega |\nabla u^{(i)}|^2\, dx + K \int_\Omega |\rho_i|^2 \, dx\,. \end{aligned}$$

Therewith, we obtain after summing over i from 1 to 2, using the ellipticity condition

(2.4) and the fact that $f^{(i)} \in L^\infty(\Omega;\mathbb{R}^3)$

$$
\begin{aligned}
&c_0 \int_\Omega |\nabla u|^2\,dx + \int_\Omega a\left|u^{(1)} - u^{(2)}\right|^2\,dx \\
&\le \int_\Omega |\rho||f||u|\,dx + \varepsilon \int_\Omega |\nabla u|^2\,dx + K\int_\Omega |\rho|^2\,dx + K \\
&\le K\|u\|_{L^6}\|\rho\|_{L^{\frac{6}{5}}} + \varepsilon \int_\Omega |\nabla u|^2\,dx + K\int_\Omega |\rho|^2\,dx + K \\
&\le K\|\nabla u\|_{L^2}\|\rho\|_{L^{\frac{6}{5}}} + \varepsilon \int_\Omega |\nabla u|^2\,dx + K\int_\Omega |\rho|^2\,dx + K \\
&\le \varepsilon\|\nabla u\|_{L^2}^2 + K\|\rho\|_{L^{\frac{6}{5}}}^2 + \varepsilon \int_\Omega |\nabla u|^2\,dx + K\int_\Omega |\rho|^2\,dx + K\,.
\end{aligned}
$$

The quantity derived from the interaction terms is positive, so we obtain after absorbing $\varepsilon\|\nabla u\|_{L^2}$ on the left-hand side the following inequality for the L^2-norm of ∇u:

$$
\tilde{c}_0 \int_\Omega |\nabla u|^2\,dx \le K\|\rho\|_{L^{\frac{6}{5}}}^2 + K\int_\Omega |\rho|^2\,dx + K\,. \tag{2.18}
$$

Now, we test equation (2.13) by ρ_i to obtain

$$
\begin{aligned}
&\sigma \int_\Omega |\nabla \rho_i|^2\,dx + \alpha_0 \int_\Omega |\rho_i|^{s+1}\,dx + \alpha \int_\Omega \rho_i^2\,dx \\
&= -\int_\Omega \operatorname{div}\left(\rho_i u^{(i)}\right)\rho_i\,dx + \frac{\alpha}{|\Omega|}\int_\Omega \rho_i\,dx\,.
\end{aligned} \tag{2.19}
$$

The second integral on the right-hand side can be absorbed on the left-hand side. The first one gives with the aid of integration by parts

$$
\begin{aligned}
\left|-\int_\Omega \operatorname{div}\left(\rho_i u^{(i)}\right)\rho_i\,dx\right| &= \left|\int_\Omega \rho_i u^{(i)}\cdot\nabla\rho_i\,dx\right| \\
&= \left|\int_\Omega u^{(i)}\cdot\nabla\left(\frac{\rho_i^2}{2}\right)\,dx\right| \\
&= \left|-\frac{1}{2}\int_\Omega \operatorname{div} u^{(i)}\rho_i^2\,dx\right| \\
&\le \varepsilon\int_\Omega |\rho_i|^4\,dx + K\int_\Omega |\nabla u^{(i)}|^2\,dx\,.
\end{aligned}
$$

Here, the first integral can be absorbed on the left-hand side of (2.19) if $s > 3$, and the second one is treated using inequality (2.18). We obtain after summing over i

$$\begin{aligned}\sigma \int_\Omega |\nabla\rho|^2\,dx + \tilde\alpha_0 \int_\Omega |\rho|^{s+1}\,dx + \tilde\alpha \int_\Omega |\rho|^2\,dx \\ \le K\left(\int_\Omega |\rho|^{6/5}\,dx\right)^{5/3} + K\int_\Omega |\rho|^2\,dx + K \\ \le \varepsilon \int_\Omega |\rho|^{s+1}\,dx + K\end{aligned}$$

by using Hölder's and Young's inequality (note that s is large enough).

Thus, we have

$$\sigma \int_\Omega |\nabla\rho|^2\,dx + \tilde\alpha_0 \int_\Omega |\rho|^{s+1}\,dx + \tilde\alpha \int_\Omega |\rho|^2\,dx \le K\,,$$

and therefore also by (2.18)

$$\int_\Omega |\nabla u|^2\,dx \le K\,.$$

So, we have coerciveness for the approximative system, and by the theory of monotone operators (cf. the articles by Višik, Leray and Lions, [Viš61], [Viš63], [LL65]) there exist weak solutions $(\rho, u) = (\rho^{\alpha_0,\alpha,\sigma}, u_{\alpha_0,\alpha,\sigma})$ to the equations (2.13)–(2.15) with

$$\begin{aligned}&\rho_i^{\alpha_0,\alpha,\sigma} \in H^1(\Omega) \cap L^{s+1}(\Omega) \text{ and}\\ &u^{(i)}_{\alpha_0,\alpha,\sigma} \in H_0^1(\Omega;\mathbb{R}^3)\,.\end{aligned}$$

The above estimates depend, of course, on α_0, α and σ.

Step 2: Nonnegativity of the densities ρ_i

Consider the following auxiliary problem

$$-\sigma\Delta G - \sum_{k=1}^{3} u_k^{(i)}\partial_k G + \alpha G + \alpha_0|\rho_i|^{s-1}G = \chi_{\{\rho_i<0\}} \quad \text{in } \Omega\,, \tag{2.20}$$

$$\nabla G\cdot\vec n = 0 \quad \text{on } \partial\Omega\,. \tag{2.21}$$

Here, $\chi_{\{\rho_i<0\}}$ denotes the characteristic function of the set $\{x|\rho_i(x)<0\}$.

Due to the weak maximum principle, the function G is nonnegative. (One can see this by firstly approximating $u^{(i)}$ by smooth functions, where $G \ge 0$ due to the classical maximum principle, and then passing to the limit, where the inequality $G \ge 0$ is conserved.)

We test by G in (2.13) to obtain

$$\int_\Omega \rho_i \chi_{\{\rho_i<0\}}\,dx \geq 0\,.$$

From this inequality we conclude that the negative part $\rho_i^- = \min(\rho_i, 0)$ of ρ_i must vanish almost everywhere in Ω, thus, $\rho_i \geq 0$ almost everywhere in Ω.

(As was remarked above, we could alternatively use a different approximation with replacing the term $\operatorname{div}(\rho_i u^{(i)})$ in the continuity equation by $\operatorname{div}(\rho_i^+ u^{(i)})$. After having proved the existence of weak solutions to this system (as in Step 1), we can test by ρ_i^- in the approximate continuity equation to see that ρ_i^- vanishes almost everywhere. Thus, we would have ensured the nonnegativity of the densities as well and could replace ρ_i^+ by ρ_i in the approximate equation.)

Since we know now that $\rho_i \geq 0$, we can replace the term $|\rho_i|^{s-1}\rho_i$ in the approximate continuity equation by ρ_i^s.

Step 3: Estimates for ρ_i and $u^{(i)}$ which are independent of α_0, α and σ

We test again by $u^{(i)}$ in (2.14)

$$\int_\Omega \sum_{k=1}^{2} L_{ik} u^{(k)} \cdot u^{(i)}\,dx = -c_i \int_\Omega u^{(i)} \cdot \nabla\rho_i\,dx + \int_\Omega \rho_i f^{(i)} \cdot u^{(i)}\,dx + \int_\Omega I^{(i)} \cdot u^{(i)}\,dx\,.$$

Since we want to derive estimates which do not depend on α_0, α and σ, we can no longer use the L^{s+1}-norm of ρ.

Thus, we cannot absorb the terms with ρ_i coming from the pressure, but we have to investigate the pressure term more closely using the approximate continutiy equation (2.13).

We get (at least formally) analogously to the theory for compressible flow for single continuum

$$\begin{aligned}
-c_i \int_\Omega u^{(i)} \cdot \nabla\rho_i\,dx &= -c_i \int_\Omega \rho_i u^{(i)} \cdot \nabla \log\rho_i\,dx \\
&= c_i \int_\Omega \operatorname{div}\left(\rho_i u^{(i)}\right) \log\rho_i\,dx \\
&= -c_i\sigma \int_\Omega \frac{|\nabla\rho_i|^2}{\rho_i}\,dx - c_i\alpha \int_\Omega \rho_i \log\rho_i\,dx \\
&\quad - c_i\alpha_0 \int_\Omega \rho_i^s \log\rho_i\,dx + c_i \frac{\alpha}{|\Omega|} \int_\Omega \log\rho_i\,dx
\end{aligned}$$

by using the equation (2.13) in the last step.

More precisely, we have to insert an auxiliary parameter $\delta > 0$ such that we can deal with ρ_i in the denominator. We obtain

$$\begin{aligned}
-c_i \int_\Omega u^{(i)} \cdot \nabla \rho_i \, dx &= -c_i \int_\Omega (\rho_i + \delta) u^{(i)} \cdot \nabla \log(\rho_i + \delta) \, dx \\
&= c_i \int_\Omega \operatorname{div} \left((\rho_i + \delta) u^{(i)} \right) \log(\rho_i + \delta) \, dx \\
&= c_i \delta \int_\Omega \operatorname{div} u^{(i)} \log(\rho_i + \delta) \, dx \\
&\quad + c_i \int_\Omega \operatorname{div} \left(\rho_i u^{(i)} \right) \log(\rho_i + \delta) \, dx \\
&= c_i \delta \int_\Omega \operatorname{div} u^{(i)} \log(\rho_i + \delta) \, dx \\
&\quad - c_i \sigma \int_\Omega \frac{|\nabla \rho_i|^2}{\rho_i + \delta} \, dx - c_i \alpha \int_\Omega \rho_i \log(\rho_i + \delta) \, dx \\
&\quad - c_i \alpha_0 \int_\Omega \rho_i^s \log(\rho_i + \delta) \, dx + c_i \frac{\alpha}{|\Omega|} \int_\Omega \log(\rho_i + \delta) \, dx
\end{aligned}$$

by using the approximate continuity equation (2.13).

It is clear that we can let the parameter δ tend to zero and that we obtain the corresponding integrals without δ after having achieved the estimates below. The first integral vanishes as $\delta \to 0$.

In the sequel we omit the technical parameter δ.

Therewith, we obtain after summing over i from 1 to 2, using the ellipticity condition (2.4) for the operators L_{ik} and absorbing the term with $\frac{\alpha}{|\Omega|}$

$$\begin{aligned}
c_0 \int_\Omega |\nabla u|^2 \, dx &+ \sum_{i=1}^{2} \left\{ c_i \sigma \int_\Omega \frac{|\nabla \rho_i|^2}{\rho_i} + c_i \tilde{\alpha} \int_\Omega \rho_i \log \rho_i \, dx + c_i \alpha_0 \int_\Omega \rho_i^s \log \rho_i \, dx \right\} \\
+ \int_\Omega a \left| u^{(1)} - u^{(2)} \right|^2 dx &\le \int_\Omega |\rho| |f| |u| \, dx + K \\
&\le K \|u\|_{L^6} \|\rho\|_{L^{\frac{6}{5}}} + K \\
&\le \varepsilon \|\nabla u\|_{L^2}^2 + K \|\rho\|_{L^{\frac{6}{5}}}^2 + K \, .
\end{aligned}$$

On the set where $\rho_i > 1$, the terms $\rho_i \log \rho_i$ and $\rho_i^s \log \rho_i$ are positive. For $\rho_i < 1$, the terms are bounded and will tend to zero as the approximation parameters tend to zero. Therefore, we can neglect the terms with α and α_0 in the limit as well as the terms with σ, which have also the right sign due to Step 2. The positive quantity from the interaction terms can be neglected as well, and, essentially, we obtain after absorbing $\varepsilon\|\nabla u\|_{L^2}$ on the left-hand side the following inequality for the L^2-norm of ∇u:

$$\tilde{c}_0 \int_\Omega |\nabla u|^2\,dx \le K\|\rho\|^2_{L^{\frac{6}{5}}} + K\,. \tag{2.22}$$

Now, we cannot continue as in the first step of the proof since we cannot use the terms with α_0 and α. Instead, we want to estimate the L^2-norm of ρ in terms of $\|\nabla u\|_{L^2}$ independently of α_0, α and σ.

Therefore, we consider the following auxiliary problem

$$\operatorname{div}\varphi^{(i)} = \rho_i - \overline{\rho_i} \quad \text{in } \Omega\,, \tag{2.23}$$

$$\varphi^{(i)} = 0 \quad \text{on } \partial\Omega\,, \tag{2.24}$$

and use $\varphi^{(i)}$ as a test function in (2.14). By φ we denote the vector $\varphi = \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix}$.

Here,

$$\overline{\rho_i} = \frac{1}{|\Omega|}\int_\Omega \rho_i\,dx$$

is the mean value of ρ_i over Ω.

The solution of the auxiliary problem is described with the help of the so-called *Bogovskii operator* introduced in [Bog80], which has to following properties (the proof of which can e.g. be found in [Gal94, p. 120ff], cf. also [NS04, p. 168ff]):

Consider the problem

$$\operatorname{div} v = f \text{ in } \Omega\,, \quad v = 0 \text{ on } \partial\Omega \tag{2.25}$$

with the compatibility condition $\int_\Omega f\,dx = 0$. Let us recall that Ω is assumed to be a Lipschitz domain.
Then there exists a "solution operator" $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)^T$ with the following properties:

- Let $W = \left\{ f \in L^p(\Omega) \,|\, \int_\Omega f\,dx = 0 \right\}$. Then the operator
$$\mathcal{B}\colon W \to W_0^{1,p}(\Omega;\mathbb{R}^3)$$
is for arbitrary $p, 1 < p < \infty$, a bounded linear operator, i.e.
$$\|\mathcal{B}[f]\|_{W_0^{1,p}(\Omega;\mathbb{R}^3)} \le C(p)\|f\|_{L^p(\Omega)}\,.$$
- The function $v = \mathcal{B}[f]$ solves (2.25).
- If f can be written in the form $f = \operatorname{div} g$ with some $g \in L^r(\Omega;\mathbb{R}^3), g\cdot\vec{n}\big|_{\partial\Omega} = 0$, then it holds
$$\|\mathcal{B}[f]\|_{L^r(\Omega)} \le C(r)\|g\|_{L^r(\Omega)}$$
for arbitrary $r, 1 < r < \infty$.

Thus, we can use the estimate

$$\|\nabla \varphi^{(i)}\|_{L^2} \le K \|\rho_i\|_{L^2} \,. \tag{2.26}$$

Testing by $\varphi^{(i)}$ in (2.14) leads to

$$\begin{aligned}
&\sum_{k=1}^{2} \left(\mu_{ik} \int_\Omega \nabla u^{(k)} : \nabla \varphi^{(i)} \, dx + (\mu_{ik} + \lambda_{ik}) \int_\Omega \operatorname{div} u^{(k)} \operatorname{div} \varphi^{(i)} \, dx \right) \\
&= c_i \int_\Omega \rho_i^2 \, dx - \overline{\rho_i} c_i \int_\Omega \rho_i \, dx + \int_\Omega \rho_i f^{(i)} \cdot \varphi^{(i)} \, dx + \int_\Omega I^{(i)} \cdot \varphi^{(i)} \, dx \,.
\end{aligned}$$

The first term on the right-hand side is the quantity we want to estimate.

Summing over i from 1 to 2 and estimating the other terms by Hölder's and Young's inequality, we obtain

$$\begin{aligned}
\int_\Omega |\rho|^2 \, dx \;\; \le \;\; & K \|\nabla u\|_{L^2}^2 + \varepsilon \|\nabla \varphi\|_{L^2}^2 + \varepsilon \|\rho\|_{L^2}^2 + K \\
& + K \|\rho\|_{L^{6/5}}^2 + K + \varepsilon \|\varphi\|_{L^6}^2 + K \|u\|_{L^2}^2 + \varepsilon \|\varphi\|_{L^2}^2 \,.
\end{aligned}$$

We make use of inequality (2.26) to estimate the terms with φ

$$\begin{aligned}
\|\nabla \varphi\|_{L^2}^2 &\le K \|\rho\|_{L^2}^2 \,, \\
\|\varphi\|_{L^6}^2 &\le K \|\nabla \varphi\|_{L^2}^2 \le K \|\rho\|_{L^2}^2 \,, \\
\|\varphi\|_{L^2}^2 &\le K \|\nabla \varphi\|_{L^2}^2 \le K \|\rho\|_{L^2}^2
\end{aligned}$$

and obtain as a result

$$\int_\Omega |\rho|^2 \, dx \le K \|\nabla u\|_{L^2}^2 + K \|\rho\|_{L^{6/5}}^2 + K \,. \tag{2.27}$$

Now, we bring the estimates together.

Using estimate (2.22) for $\|\nabla u\|_{L^2}$ in (2.27) leads to

$$\int_\Omega |\rho|^2 \, dx \le K \|\rho\|_{L^{6/5}}^2 + K \,.$$

Testing by 1 in equation (2.13) (which has a solution according to Step 1) and neglecting the term $\alpha_0 \int_\Omega \rho_i^s \, dx$, which has according to Step 2 the right sign, gives $\int_\Omega \rho_i \, dx \le 1$ from the $\alpha \rho_i$ term. This is also conserved in the limit, and thus, we can treat $\|\rho\|_{L^{6/5}}$ by interpolation

$$\|\rho\|_{L^{\frac{6}{5}}} \le \|\rho\|_{L^2}^{\frac{1}{3}} \|\rho\|_{L^1}^{\frac{2}{3}} \le \|\rho\|_{L^2}^{\frac{1}{3}}$$

and obtain

$$\int_\Omega |\rho|^2\,dx \le K\|\rho\|_{L^2}^{\frac{2}{3}} + K\,.$$

After applying Young's inequality to $\|\rho\|_{L^2}^{\frac{2}{3}}$ and absorbing the L^2-norm of ρ on the left-hand side, we have

$$\int_\Omega |\rho|^2\,dx \le K\,.$$

From (2.22) it follows also that

$$\int_\Omega |\nabla u|^2\,dx \le K\,.$$

Now, we have shown that the solutions fulfill estimates which are independent of α_0, α and σ:

$$\|\rho^{\alpha_0,\alpha,\sigma}\|_{L^2} \le K \text{ and } \|u_{\alpha_0,\alpha,\sigma}\|_{H_0^1} \le K\,,$$

and the proof of Lemma 2.1 is completed. □

Now, we want to prove Theorem 2.1.
We consider first the limit process as $\alpha_0 \to 0$. Due to the estimates from Lemma 2.1, we can extract subsequences, again denoted by $\rho_i^{\alpha_0,\alpha,\sigma}, u^{(i)}_{\alpha_0,\alpha,\sigma}$ such that for $i = 1, 2$

$$\begin{aligned}
\rho_i^{\alpha_0,\alpha,\sigma} &\rightharpoonup \rho_i^{\alpha,\sigma} \text{ weakly in } L^2\,,\\
u^{(i)}_{\alpha_0,\alpha,\sigma} &\rightharpoonup u^{(i)}_{\alpha,\sigma} \text{ weakly in } H_0^1, \text{ and, owing to Rellich–Kondrashov's theorem,}\\
u^{(i)}_{\alpha_0,\alpha,\sigma} &\to u^{(i)}_{\alpha,\sigma} \text{ strongly in } L^q, q \in [1,6)\,, \text{ as } \alpha_0 \to 0\,.
\end{aligned}$$

We can pass to the limit in the equations (2.13)–(2.14) (in the weak sense). Since there are no nonlinear terms with respect to ρ in the equations (2.1), (2.2), weak convergence of the densities is sufficient to pass to the limit.

The limits $\rho_i^{\alpha,\sigma}$, $u^{(i)}_{\alpha,\sigma}$ solve the equations

$$-\sigma\Delta\rho_i^{\alpha,\sigma} + \operatorname{div}\left(\rho_i^{\alpha,\sigma}u^{(i)}_{\alpha,\sigma}\right) + \alpha\rho_i^{\alpha,\sigma} = \frac{\alpha}{|\Omega|} \quad \text{in } \Omega\,, \tag{2.28}$$

$$\sum_{k=1}^{2} L_{ik}u^{(k)}_{\alpha,\sigma} = -c_i\nabla\rho_i^{\alpha,\sigma} + \rho_i^{\alpha,\sigma}f^{(i)} + (-1)^{i+1}a\left(u^{(2)}_{\alpha,\sigma} - u^{(1)}_{\alpha,\sigma}\right) \text{ in } \Omega\,, \tag{2.29}$$

$$u^{(i)}_{\alpha,\sigma} = 0\,, \qquad \nabla\rho_i^{\alpha,\sigma}\cdot\vec{n} = 0 \quad \text{on } \partial\Omega\,, \tag{2.30}$$

and fulfill the estimates

$$\|\rho^{\alpha,\sigma}\|_{L^2} \le K \text{ and } \|u_{\alpha,\sigma}\|_{H_0^1} \le K \tag{2.31}$$

uniformly in α and σ.
Moreover, we know due to Lemma 2.1 that $\rho_i^{\alpha,\sigma} \geq 0$ for $i = 1, 2$ since we can simply pass to the limit in this inequality as $\alpha_0 \to 0$.

Testing by 1 in equation (2.28), which is fulfilled by the limit functions $\rho_i^{\alpha,\sigma}, u_{\alpha,\sigma}^{(i)}$, we obtain for $i = 1, 2$ the equation

$$\int_\Omega \rho_i^{\alpha,\sigma}\, dx = 1\,. \tag{2.32}$$

Thus, everything is prepared for the passage to the limit as $\alpha, \sigma \to 0$. Using the estimates (2.31), we can choose a subsequence, again denoted by $\rho_i^{\alpha,\sigma}, u_{\alpha,\sigma}^{(i)}$, such that for $\alpha, \sigma \to 0$

$$\begin{aligned} \rho_i^{\alpha,\sigma} &\rightharpoonup \rho_i \text{ weakly in } L^2\,, \\ u_{\alpha,\sigma}^{(i)} &\rightharpoonup u^{(i)} \text{ weakly in } H_0^1, \text{ and, owing to the compact embedding,} \\ u_{\alpha,\sigma}^{(i)} &\to u^{(i)} \text{ strongly in } L^q, q \in [1, 6)\,. \end{aligned}$$

This is sufficient to pass to the limit in the equations (2.28)–(2.30). The limits $u^{(i)} \in H_0^1(\Omega; \mathbb{R}^3)$, $\rho_i \in L^2(\Omega)$ fulfill the equations (2.1)–(2.2), (2.5). Moreover, passing to the limit in equation (2.32) gives that

$$\int_\Omega \rho_i\, dx = 1$$

for $i = 1, 2$, i.e. also equation (2.3) is fulfilled. Furthermore, the property $\rho_i \geq 0$ is conserved in the limit, and Theorem 2.1 is proven.

2.2 Regularity

In this section we prove some additional local regularity properties for the densities ρ_i and the velocity gradients $\nabla u^{(i)}$, where the solution (ρ, u) is obtained as the limit of the approximative problem (2.13)–(2.15) as in the previous section. In fact, we obtain estimates for ρ_i and $\nabla u^{(i)}$ in L_{loc}^p for all p with $1 \leq p < \infty$.

For this purpose, we need an additional condition on the pressure law which reads as follows: for $m \geq 1$ and for all $\rho_1, \rho_2 \geq 0$

$$(\rho_1^m, \rho_2^m)\, A_0 \begin{pmatrix} P_1(\rho) \\ P_2(\rho) \end{pmatrix} \geq C_0 |\rho|^{m+1} - K\,. \tag{2.33}$$

Here, the matrix A_0, which comes into play by the equation for the effective viscous flux, is defined by

$$A_0 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} := \begin{pmatrix} \beta_0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2\mu_{11} + \lambda_{11} & 2\mu_{12} + \lambda_{12} \\ 2\mu_{21} + \lambda_{21} & 2\mu_{22} + \lambda_{22} \end{pmatrix}^{-1}.$$

Condition (2.33) holds under reasonable assumptions on the coefficients c_i and the entries a_{ij} of the matrix A_0 for the pressure law $P_i(\rho) = c_i\rho_i$.

If we impose for instance

$$|a_{12}| < \frac{c_1}{c_2}a_{11} \text{ and } |a_{21}| < \frac{c_2}{c_1}a_{22} \tag{2.34}$$

and $a_{11}c_1 = a_{22}c_2$,[1] we can prove that condition (2.33) holds.

In fact, we calculate with the aid of Young's inequality

$$\begin{aligned}
&(\rho_1^m, \rho_2^m)\, A_0 \begin{pmatrix} c_1\rho_1 \\ c_2\rho_2 \end{pmatrix} \\
&= a_{11}c_1\rho_1^{m+1} + a_{12}c_2\rho_1^m\rho_2 + a_{21}c_1\rho_1\rho_2^m + a_{22}c_2\rho_2^{m+1} \\
&= a_{11}c_1\rho_1^{m+1} + a_{11}c_1\frac{a_{12}c_2}{a_{11}c_1}\rho_1^m\rho_2 + a_{22}c_2\frac{a_{21}c_1}{a_{22}c_2}\rho_1\rho_2^m + a_{22}c_2\rho_2^{m+1} \\
&\geq a_{11}c_1\rho_1^{m+1} - \frac{m}{m+1}a_{11}c_1\rho_1^{m+1} - \frac{1}{m+1}a_{11}c_1\left(\frac{a_{12}c_2}{a_{11}c_1}\right)^{m+1}\rho_2^{m+1} \\
&+a_{22}c_2\rho_2^{m+1} - \frac{m}{m+1}a_{22}c_2\rho_2^{m+1} - \frac{1}{m+1}a_{22}c_2\left(\frac{a_{21}c_1}{a_{22}c_2}\right)^{m+1}\rho_1^{m+1} \\
&> \frac{1}{m+1}\left(a_{11}c_1\rho_1^{m+1} + a_{22}c_2\rho_2^{m+1}\right) - \frac{1}{m+1}\left(a_{11}c_1\rho_2^{m+1} + a_{22}c_2\rho_1^{m+1}\right),
\end{aligned}$$

where we used the condition (2.34) in the last step. The last strict inequality holds for $|\rho| = 1$. Using the condition $a_{11}c_1 = a_{22}c_2$, we have proved that

$$(\rho_1^m, \rho_2^m)\, A_0 \begin{pmatrix} c_1\rho_1 \\ c_2\rho_2 \end{pmatrix} > 0 \text{ on the set } (|\rho| = 1)\,.$$

From this inequality the coerciveness condition (2.33) follows.

For $m = 1$ the condition (2.33) corresponds to the positive definiteness of the matrix $A_0\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$. We can guarantee the positive definiteness of $A_0\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$ by choosing the parameter β_0 as $\frac{c_1}{c_2}$, as follows from

$$\begin{aligned}
&(x_1, x_2)\begin{pmatrix} \beta_0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2\mu_{11}+\lambda_{11} & 2\mu_{12}+\lambda_{12} \\ 2\mu_{21}+\lambda_{21} & 2\mu_{22}+\lambda_{22} \end{pmatrix}^{-1}\begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= (\beta_0 x_1, x_2)\begin{pmatrix} 2\mu_{11}+\lambda_{11} & 2\mu_{12}+\lambda_{12} \\ 2\mu_{21}+\lambda_{21} & 2\mu_{22}+\lambda_{22} \end{pmatrix}^{-1}\begin{pmatrix} c_1x_1 \\ c_2x_2 \end{pmatrix} \\
&= c_2(\beta_0 x_1, x_2)\begin{pmatrix} 2\mu_{11}+\lambda_{11} & 2\mu_{12}+\lambda_{12} \\ 2\mu_{21}+\lambda_{21} & 2\mu_{22}+\lambda_{22} \end{pmatrix}^{-1}\begin{pmatrix} \frac{c_1}{c_2}x_1 \\ x_2 \end{pmatrix}
\end{aligned}$$

[1]This is equivalent in terms of the viscosities to $c_1\beta_0(2\mu_{22}+\lambda_{22}) > |\beta_0c_2(2\mu_{12}+\lambda_{12})|$ and $c_2(2\mu_{11}+\lambda_{11}) > |c_1(2\mu_{21}+\lambda_{21})|$, and $c_1\beta_0(2\mu_{22}+\lambda_{22}) = c_2(2\mu_{11}+\lambda_{11})$.

and the positive definiteness of the matrix $\begin{pmatrix} 2\mu_{11}+\lambda_{11} & 2\mu_{12}+\lambda_{12} \\ 2\mu_{21}+\lambda_{21} & 2\mu_{22}+\lambda_{22} \end{pmatrix}^{-1}$.

If we assume the coerciveness condition, we can prove the following

Theorem 2.2 *Let $P(\rho) = (c_1\rho_1, c_2\rho_2)^T$ satisfy the coerciveness condition (2.33). Then we have for a solution $(\rho, u), \rho = (\rho_1, \rho_2)^T, u = (u^{(1),T}, u^{(2),T})^T$, of the equations (2.1)–(2.3), (2.5), obtained as the limit of solutions of the approximate problem (2.13)–(2.15), for $i = 1, 2$*

$$\rho_i \in L^p_{loc}(\Omega)\,, \quad \nabla u^{(i)} \in L^p_{loc}(\Omega; \mathbb{R}^{3\times 3}) \text{ for all } p\,, 1 \le p < \infty\,.$$

In order to prove this theorem we consider again solutions of the approximate system (2.13)–(2.15) and prove

Theorem 2.3 *Let $(\rho^{\alpha_0,\alpha,\sigma}, u_{\alpha_0,\alpha,\sigma})$ be solutions to the approximate system (2.13)–(2.15) with the pressure P satisfying the condition (2.33). Then for all p with $1 \le p < \infty$*

$$\rho_i^{\alpha_0,\alpha,\sigma} \in L^p_{loc}(\Omega), \nabla u^{(i)}_{\alpha_0,\alpha,\sigma} \in L^p_{loc}(\Omega; \mathbb{R}^{3\times 3})$$

and

$$\|\rho_i^{\alpha_0,\alpha,\sigma}\|_{L^p_{loc}} \le K, \|\nabla u^{(i)}_{\alpha_0,\alpha,\sigma}\|_{L^p_{loc}} \le K \text{ uniformly in } \alpha_0, \alpha \text{ and } \sigma\,.$$

Proof:
The proof is split into two parts. In the first part we prove with the aid of a bootstrap argument that $\rho_i^{\alpha_0,\alpha,\sigma} \in L^6_{loc}(\Omega)$ and $\nabla u^{(i)}_{\alpha_0,\alpha,\sigma} \in L^6_{loc}(\Omega; \mathbb{R}^{3\times 3})$ with estimates independent of α_0, α and σ. In the second part we utilize these estimates to prove uniform estimates for $\rho_i^{\alpha_0,\alpha,\sigma}$ and $\nabla u^{(i)}_{\alpha_0,\alpha,\sigma}$ in all L^p_{loc}, where we again use a bootstrap argument. In both parts the key tool is the so-called equation for the effective viscous flux.
(It would be possible to perform the proof in one step, but we prefer this presentation to make the arguments clearer.)

Part I: We show that $\rho_i^{\alpha_0,\alpha,\sigma} \in L^6_{loc}(\Omega), \nabla u^{(i)}_{\alpha_0,\alpha,\sigma} \in L^6_{loc}(\Omega; \mathbb{R}^{3\times 3})$ with $\|\rho_i^{\alpha_0,\alpha,\sigma}\|_{L^6_{loc}} \le K, \|\nabla u^{(i)}_{\alpha_0,\alpha,\sigma}\|_{L^6_{loc}} \le K$ uniformly in α_0, α and σ.

In order to prove these estimates we have to use an equation which we call in analogy to the classical case of the Navier–Stokes equations for *one* compressible fluid the *equation for the effective viscous flux* or *effective pressure.* To derive this equation, we test equation (2.14) by $\nabla(\tau\varphi^{(i)}), i = 1, 2$, wherein $\varphi^{(i)}$ will be chosen later on and

τ is a smooth localization function with $\tau = 1$ inside Ω and $\tau = 0$ at the boundary:

$$\begin{aligned}
&\int_\Omega \begin{pmatrix} \Delta(\tau\varphi^{(1)}) \\ \Delta(\tau\varphi^{(2)}) \end{pmatrix}^T \begin{pmatrix} 2\mu_{11}+\lambda_{11} & 2\mu_{12}+\lambda_{12} \\ 2\mu_{21}+\lambda_{21} & 2\mu_{22}+\lambda_{22} \end{pmatrix} \begin{pmatrix} \operatorname{div} u^{(1)}_{\alpha_0,\alpha,\sigma} \\ \operatorname{div} u^{(2)}_{\alpha_0,\alpha,\sigma} \end{pmatrix} dx \\
= &\int_\Omega \begin{pmatrix} \Delta(\tau\varphi^{(1)}) \\ \Delta(\tau\varphi^{(2)}) \end{pmatrix}^T \begin{pmatrix} P_1(\rho^{\alpha_0,\alpha,\sigma}) \\ P_2(\rho^{\alpha_0,\alpha,\sigma}) \end{pmatrix} dx \\
&- \int_\Omega \begin{pmatrix} \Delta(\tau\varphi^{(1)}) \\ \Delta(\tau\varphi^{(2)}) \end{pmatrix}^T \begin{pmatrix} \operatorname{div} \Delta^{-1}(\rho_1^{\alpha_0,\alpha,\sigma} f^{(1)} + I^{(1)}_{\alpha_0,\alpha,\sigma}) \\ \operatorname{div} \Delta^{-1}(\rho_2^{\alpha_0,\alpha,\sigma} f^{(2)} + I^{(2)}_{\alpha_0,\alpha,\sigma}) \end{pmatrix} dx\,.
\end{aligned}$$

Here, we understand Δ^{-1} as solving the Laplace equation in $\mathbb{R}^3$. The functions under consideration are extended by zero outside of the domain Ω.

We used the identities

$$\int_\Omega \rho_i^{\alpha_0,\alpha,\sigma} f^{(i)} \cdot \nabla \left(\tau\varphi^{(i)}\right) dx = -\int_\Omega \operatorname{div} \Delta^{-1} \left(\rho_i^{\alpha_0,\alpha,\sigma} f^{(i)}\right) \Delta \left(\tau\varphi^{(i)}\right) dx$$

and

$$\int_\Omega I^{(i)}_{\alpha_0,\alpha,\sigma} \cdot \nabla \left(\tau\varphi^{(i)}\right) dx = -\int_\Omega \operatorname{div} \Delta^{-1} \left(I^{(i)}_{\alpha_0,\alpha,\sigma}\right) \Delta \left(\tau\varphi^{(i)}\right) dx\,.$$

These equations are valid for smooth functions $\tau\varphi^{(i)}$ and by density arguments also for functions such that $\Delta\left(\tau\varphi^{(i)}\right) \in L^2$. (Observe that $\tau\varphi^{(i)}$ with $\Delta\left(\tau\varphi^{(i)}\right) \in L^2$ needs not to be bounded. Nevertheless, the density argument works since only $\nabla\left(\tau\varphi^{(i)}\right)$ occurs in the formulae, not $\tau\varphi^{(i)}$ itself.) In fact, by denoting $h_i = \Delta^{-1}\left(\rho_i^{\alpha_0,\alpha,\sigma} f^{(i)}\right)$:

$$\begin{aligned}
\left|\int_\Omega \operatorname{div} h_i \Delta\left(\tau\varphi^{(i)}\right) dx\right| &\le \|\Delta\left(\tau\varphi^{(i)}\right)\|_{L^2} \|\operatorname{div} h_i\|_{L^2} \\
&\le c\|\Delta\left(\tau\varphi^{(i)}\right)\|_{L^2} \|\rho_i^{\alpha_0,\alpha,\sigma} f^{(i)}\|_{(H_0^1)^*} \\
&\le c\|\Delta\left(\tau\varphi^{(i)}\right)\|_{L^2} \|\rho_i^{\alpha_0,\alpha,\sigma} f^{(i)}\|_{L^{\frac{6}{5}}} \\
&\le c\|\Delta\left(\tau\varphi^{(i)}\right)\|_{L^2} \|\rho_i^{\alpha_0,\alpha,\sigma}\|_{L^{s+1}} \|f^{(i)}\|_{L^{\frac{6(s+1)}{5s-1}}} \\
&\le c\|\Delta\left(\tau\varphi^{(i)}\right)\|_{L^2}\,.
\end{aligned}$$

By setting

$$A_0 = \begin{pmatrix} \beta_0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2\mu_{11}+\lambda_{11} & 2\mu_{12}+\lambda_{12} \\ 2\mu_{21}+\lambda_{21} & 2\mu_{22}+\lambda_{22} \end{pmatrix}^{-1}$$

and choosing $\varphi = A_0^T \tilde{\psi}, \varphi = \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix}, \tilde{\psi} = \begin{pmatrix} \tilde{\psi}^{(1)} \\ \tilde{\psi}^{(2)} \end{pmatrix}$, this equation becomes

$$\begin{aligned} &\int_\Omega \begin{pmatrix} \Delta(\tau\tilde{\psi}^{(1)}) \\ \Delta(\tau\tilde{\psi}^{(2)}) \end{pmatrix}^T \begin{pmatrix} \beta_0 \operatorname{div} u^{(1)}_{\alpha_0,\alpha,\sigma} \\ \operatorname{div} u^{(2)}_{\alpha_0,\alpha,\sigma} \end{pmatrix} dx \\ &= \int_\Omega \begin{pmatrix} \Delta(\tau\tilde{\psi}^{(1)}) \\ \Delta(\tau\tilde{\psi}^{(2)}) \end{pmatrix}^T A_0 \begin{pmatrix} P_1(\rho^{\alpha_0,\alpha,\sigma}) \\ P_2(\rho^{\alpha_0,\alpha,\sigma}) \end{pmatrix} dx \\ &- \int_\Omega \begin{pmatrix} \Delta(\tau\tilde{\psi}^{(1)}) \\ \Delta(\tau\tilde{\psi}^{(2)}) \end{pmatrix}^T A_0 \begin{pmatrix} \operatorname{div} \Delta^{-1}(\rho_1^{\alpha_0,\alpha,\sigma} f^{(1)} + I^{(1)}_{\alpha_0,\alpha,\sigma}) \\ \operatorname{div} \Delta^{-1}(\rho_2^{\alpha_0,\alpha,\sigma} f^{(2)} + I^{(2)}_{\alpha_0,\alpha,\sigma}) \end{pmatrix} dx\,. \end{aligned}$$

By separating we obtain for $i = 1, 2$ the following equation with a function $\psi^{(i)}$, which can be chosen suitably,

$$\begin{aligned} \int_\Omega \hat{\beta}_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \Delta(\tau\psi^{(i)})\, dx &= \int_\Omega (A_0 P(\rho^{\alpha_0,\alpha,\sigma}))_i\, \Delta(\tau\psi^{(i)})\, dx \\ &- \int_\Omega \left(A_0 \operatorname{div} \Delta^{-1}\left(\rho^{\alpha_0,\alpha,\sigma} f + I_{\alpha_0,\alpha,\sigma}\right)\right)_i \Delta(\tau\psi^{(i)})\, dx\,, \end{aligned} \tag{2.35}$$

whereby we use the notation

$$\hat{\beta}_i = \begin{cases} \beta_0\,, & i = 1\,, \\ 1\,, & i = 2\,, \end{cases}$$

$$P(\rho^{\alpha_0,\alpha,\sigma}) = \begin{pmatrix} c_1 \rho_1^{\alpha_0,\alpha,\sigma} \\ c_2 \rho_2^{\alpha_0,\alpha,\sigma} \end{pmatrix} \quad \text{and}$$

$$\operatorname{div} \Delta^{-1}\left(\rho^{\alpha_0,\alpha,\sigma} f + I_{\alpha_0,\alpha,\sigma}\right) = \begin{pmatrix} \operatorname{div} \Delta^{-1}\left(\rho_1^{\alpha_0,\alpha,\sigma} f^{(1)} + I^{(1)}_{\alpha_0,\alpha,\sigma}\right) \\ \operatorname{div} \Delta^{-1}\left(\rho_2^{\alpha_0,\alpha,\sigma} f^{(2)} + I^{(2)}_{\alpha_0,\alpha,\sigma}\right) \end{pmatrix}.$$

With equation (2.35) we have derived the so-called equation for the effective viscous flux. In the sequel, we will make use of this equation quite frequently.

In order to perform a bootstrap argument we choose now a special function $\psi^{(i)}$. In fact, we solve

$$\Delta\psi^{(i)} = (\rho_i^{\alpha_0,\alpha,\sigma})^{m_j}\ \tau$$

in $\mathbb{R}^3$, where the functions under consideration are extended by zero outside Ω and $\tau \in \mathcal{D}(\Omega)$ is a smooth localization function, $m_j \in \mathbb{R}^+$.

In order to ensure that the corresponding terms with $\nabla\psi^{(i)}$ in (2.35) are defined, we have to choose m_j in such a way that

$$(\rho_i^{\alpha_0,\alpha,\sigma})^{m_j} \in L^{6/5}_{loc}(\Omega)\,,$$

where the estimate should not depend on α_0, α and σ. According to the definition of $\psi^{(i)}$ we know that $\nabla\psi^{(i)} \in L^2_{loc}(\Omega;\mathbb{R}^3)$, and the integrals with $\operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma}$ and $\nabla\psi^{(i)}$ in equation (2.35) make sense.

In the first section of this chapter we have proved that solutions (ρ, u) with $\rho_i \in L^2(\Omega)$ exist. As we would like to use estimates which are independent of the approximation parameters α_0, α and σ, we can utilize in the first step of this bootstrap argument only the estimate $\|\rho_i^{\alpha_0,\alpha,\sigma}\|_{L^2} \leq K$. Thus, we have to choose

$$m_0 = \frac{5}{3}.$$

Then, by using the coerciveness condition for the pressure law, we derive from equation (2.35) an estimate $\|\rho_i^{\alpha_0,\alpha,\sigma}\|_{L^{m_0+1}_{loc}} = \|\rho_i^{\alpha_0,\alpha,\sigma}\|_{L^{8/3}_{loc}} \leq K$ and we can choose in the second step

$$m_1 = \frac{20}{9}.$$

In general, we obtain a sequence of $m_j, j = 0, 1, 2, \ldots$, which is defined recursively by

$$m_0 = \frac{5}{3}, \qquad m_{j+1} = \frac{5}{6}\left(m_j + 1\right).$$

This sequence is monotone and bounded and, thus, convergent. Its limit is 5, such that we can expect in the end an estimate for $\rho_i^{\alpha_0,\alpha,\sigma}$ in $L^6_{loc}(\Omega)$ with $\|\rho_i^{\alpha_0,\alpha,\sigma}\|_{L^6_{loc}} \leq K$ uniformly and therefore also $\nabla u^{(i)}_{\alpha_0,\alpha,\sigma} \in L^6_{loc}(\Omega;\mathbb{R}^{3\times3})$ with $\|\nabla u^{(i)}_{\alpha_0,\alpha,\sigma}\|_{L^6_{loc}} \leq K$ uniformly in α_0, α and σ.

We derive now the estimate from equation (2.35).

At first, we investigate the terms coming from $\operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma}$.

$$\int_\Omega \hat{\beta}_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \Delta\left(\tau\psi^{(i)}\right)\, dx = \int_\Omega \hat{\beta}_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \Delta\tau\psi^{(i)}\, dx$$
$$+2\int_\Omega \hat{\beta}_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \nabla\tau\cdot\nabla\psi^{(i)}\, dx + \int_\Omega \hat{\beta}_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \tau\Delta\psi^{(i)}\, dx\,.$$

Due to $\nabla u^{(i)}_{\alpha_0,\alpha,\sigma} \in L^2(\Omega;\mathbb{R}^{3\times3})$ and the choice of $\psi^{(i)}$ with $\nabla\psi^{(i)} \in L^2_{loc}(\Omega;\mathbb{R}^3)$, the first two integrals are uniformly bounded. We investigate now the third one. This term is defined in the first place by using qualitatively estimates for fixed $\alpha_0 > 0$ and $\sigma > 0$. If we choose s large, we can assure that $\rho_i^{\alpha_0,\alpha,\sigma} \in L^{3+\delta}(\Omega), \delta > 0$, and thus also $P_i(\rho^{\alpha_0,\alpha,\sigma}) \in L^{3+\delta}(\Omega)$ for $\alpha_0 > 0$ fix. From equation (2.14) it follows that $\nabla u^{(i)}_{\alpha_0,\alpha,\sigma} \in L^{3+\delta}(\Omega;\mathbb{R}^{3\times3})$ as well, and then by the embedding theorems $u^{(i)}_{\alpha_0,\alpha,\sigma} \in L^\infty(\Omega;\mathbb{R}^3)$. For $\sigma > 0$ fix it follows then also that $\rho_i^{\alpha_0,\alpha,\sigma} \in L^\infty(\Omega)$. Thus, the third integral is defined and gives with the aid of integration by parts

$$
\begin{aligned}
&\int_\Omega \hat{\beta}_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \tau \Delta\psi^{(i)}\, dx \\
&\quad = \int_\Omega \hat{\beta}_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \tau^2 \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_j}\, dx \\
&\quad = -\int_\Omega \hat{\beta}_i u^{(i)}_{\alpha_0,\alpha,\sigma} \cdot \nabla\tau^2 \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_j}\, dx \\
&\qquad - \int_\Omega \hat{\beta}_i \rho_i^{\alpha_0,\alpha,\sigma} u^{(i)}_{\alpha_0,\alpha,\sigma} \cdot \nabla \left(\left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_j}\right) \frac{1}{\rho_i^{\alpha_0,\alpha,\sigma}} \tau^2\, dx \\
&\quad = -\int_\Omega \hat{\beta}_i u^{(i)}_{\alpha_0,\alpha,\sigma} \cdot \nabla\tau^2 \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_j}\, dx \\
&\qquad - \int_\Omega \hat{\beta}_i \rho_i^{\alpha_0,\alpha,\sigma} u^{(i)}_{\alpha_0,\alpha,\sigma} \cdot \nabla \left(\frac{m_j}{m_j-1} \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_j-1}\right) \tau^2\, dx\,.
\end{aligned}
$$

The first integral is bounded since $u^{(i)}_{\alpha_0,\alpha,\sigma} \in L^6(\Omega;\mathbb{R}^3)\,, \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_j} \in L^{6/5}_{loc}(\Omega)$ and $\nabla\tau^2 \in L^\infty_{loc}(\Omega;\mathbb{R}^3)$.

With the aid of integration by parts we obtain from the second one

$$
\begin{aligned}
&-\int_\Omega \hat{\beta}_i \rho_i^{\alpha_0,\alpha,\sigma} u^{(i)}_{\alpha_0,\alpha,\sigma} \cdot \nabla \left(\frac{m_j}{m_j-1} \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_j-1}\right) \tau^2\, dx \\
&\quad = \int_\Omega \hat{\beta}_i \operatorname{div}\left(\rho_i^{\alpha_0,\alpha,\sigma} u^{(i)}_{\alpha_0,\alpha,\sigma}\right) \frac{m_j}{m_j-1} \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_j-1} \tau^2\, dx \\
&\qquad + \int_\Omega \hat{\beta}_i \frac{m_j}{m_j-1} \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_j} u^{(i)}_{\alpha_0,\alpha,\sigma} \cdot \nabla\tau^2\, dx\,.
\end{aligned}
$$

The second integral is bounded because $\left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_j} \in L^{6/5}_{loc}(\Omega)$ and $u^{(i)}_{\alpha_0,\alpha,\sigma} \in L^6(\Omega;\mathbb{R}^3)$. The first one is treated by using the approximate continuity equation (2.13).

With $\rho_i^{\alpha_0,\alpha,\sigma}$ and $u^{(i)}_{\alpha_0,\alpha,\sigma}$ extended by zero outside Ω, it holds

$$
-\sigma\Delta\rho_i^{\alpha_0,\alpha,\sigma} + \operatorname{div}\left(\rho_i^{\alpha_0,\alpha,\sigma} u^{(i)}_{\alpha_0,\alpha,\sigma}\right) + \alpha_0 \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^s + \alpha\rho_i^{\alpha_0,\alpha,\sigma} = \frac{\alpha}{|\Omega|} \text{ in } \Omega\,,
$$

and due to the boundary conditions $u^{(i)}_{\alpha_0,\alpha,\sigma}\big|_{\partial\Omega} = 0$, we have

$$
\operatorname{div}\left(\rho_i^{\alpha_0,\alpha,\sigma} u^{(i)}_{\alpha_0,\alpha,\sigma}\right) = 0 \text{ in } \mathbb{R}^3 \setminus \Omega\,.
$$

We obtain

$$\begin{aligned}
&\int_\Omega \hat{\beta}_i \operatorname{div}\left(\rho_i^{\alpha_0,\alpha,\sigma} u^{(i)}_{\alpha_0,\alpha,\sigma}\right) \frac{m_j}{m_j-1} \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_j-1} \tau^2\, dx \\
&= -\sigma \int_\Omega \hat{\beta}_i \left|\nabla \rho_i^{\alpha_0,\alpha,\sigma}\right|^2 m_j \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_j-2}\ \tau^2\, dx \\
&\quad -\sigma \int_\Omega \hat{\beta}_i \nabla \rho_i^{\alpha_0,\alpha,\sigma} \cdot \nabla \tau^2 \frac{m_j}{m_j-1} \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_j-1}\ dx \\
&\quad -\alpha_0 \int_\Omega \hat{\beta}_i \frac{m_j}{m_j-1} \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{s+m_j-1} \tau^2\, dx - \alpha \int_\Omega \hat{\beta}_i \frac{m_j}{m_j-1} \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_j} \tau^2\, dx \\
&\quad + \frac{\alpha}{|\Omega|} \int_\Omega \hat{\beta}_i \frac{m_j}{m_j-1} \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_j-1}\ \tau^2\, dx\,.
\end{aligned}$$

If we choose s large enough such that for fixed $\alpha_0 > 0$ the density and therewith the pressure $P_i(\rho^{\alpha_0,\alpha,\sigma}) \in L^{3+\delta}_{loc}(\Omega)$, we can conclude via equation (2.14) that also $\nabla u^{(i)}_{\alpha_0,\alpha,\sigma} \in L^{3+\delta}_{loc}(\Omega;\mathbb{R}^{3\times 3})$ and by the embedding theorems $u^{(i)}_{\alpha_0,\alpha,\sigma} \in L^\infty_{loc}(\Omega;\mathbb{R}^3)$. Then also $\rho_i^{\alpha_0,\alpha,\sigma} \in L^\infty_{loc}(\Omega)$ for $\sigma > 0$ fixed.
Thus, all the above integrals are bounded for $\alpha_0 > 0, \alpha > 0, \sigma > 0$ fixed and tend to zero as $\alpha_0 \to 0$ and $\alpha \to 0, \sigma \to 0$.

So all terms coming from $\operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma}$ in equation (2.35) are bounded uniformly in α_0, α and σ.

The terms with f and $I_{\alpha_0,\alpha,\sigma}$ in (2.35), which are of lower order, are bounded as well such that there remains for $i = 1, 2$

$$\int_\Omega \left(A_0 P(\rho^{\alpha_0,\alpha,\sigma})\right)_i \Delta\left(\tau \psi^{(i)}\right)\ dx \le K\,.$$

We calculate

$$\begin{aligned}
\int_\Omega \left(A_0 P(\rho^{\alpha_0,\alpha,\sigma})\right)_i \Delta\left(\tau \psi^{(i)}\right)\ dx &= \int_\Omega \left(A_0 P(\rho^{\alpha_0,\alpha,\sigma})\right)_i \Delta \tau \psi^{(i)}\, dx \\
+2 \int_\Omega \left(A_0 P(\rho^{\alpha_0,\alpha,\sigma})\right)_i \nabla \tau \cdot \nabla \psi^{(i)}\, dx &+ \int_\Omega \left(A_0 P(\rho^{\alpha_0,\alpha,\sigma})\right)_i \tau \Delta \psi^{(i)}\, dx\,.
\end{aligned}$$

The first two integrals are bounded because $\rho_i^{\alpha_0,\alpha,\sigma} \in L^2(\Omega)$ and $\nabla \psi^{(i)} \in L^2_{loc}(\Omega;\mathbb{R}^3)$ due to the choice of $\psi^{(i)}$.

To summarize, we have estimated the integral

$$\int_\Omega \left(A_0 P(\rho_i^{\alpha_0,\alpha,\sigma})\right)_i \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_j}\ \tau^2\, dx \le K\,.$$

Summing over i from 1 to 2 and using the coerciveness condition (2.33) leads to

$$\int_\Omega \left|\rho^{\alpha_0,\alpha,\sigma}\right|^{m_j+1}\ \tau^2\, dx \le K\,,$$

i.e.

$$\rho_i^{\alpha_0,\alpha,\sigma} \in L_{loc}^{m_j+1}(\Omega)\,.$$

As explained above, due to the possible choices of m_j, we obtain in the end

$$\rho_i^{\alpha_0,\alpha,\sigma} \in L_{loc}^{6}(\Omega)\,,$$

and by equation (2.14) also

$$\nabla u_{\alpha_0,\alpha,\sigma}^{(i)} \in L_{loc}^{6}(\Omega;\mathbb{R}^{3\times 3})$$

with estimates which are uniform in α_0, α and σ.

Part II: In this part we use the estimates which we have just obtained to prove that we can get estimates in all L_{loc}^p, $1 \le p < \infty$, for $\rho_i^{\alpha_0,\alpha,\sigma}$ and $\nabla u_{\alpha_0,\alpha,\sigma}^{(i)}$.

We consider again the equation for the effective viscous flux

$$\begin{aligned}\int_\Omega \hat\beta_i \operatorname{div} u_{\alpha_0,\alpha,\sigma}^{(i)} \Delta(\tau\psi^{(i)})\,dx &= \int_\Omega \left(A_0 P(\rho^{\alpha_0,\alpha,\sigma})\right)_i \Delta(\tau\psi^{(i)})\,dx \\ &- \int_\Omega \left(A_0 \operatorname{div}\Delta^{-1}\left(\rho^{\alpha_0,\alpha,\sigma} f + I_{\alpha_0,\alpha,\sigma}\right)\right)_i \Delta(\tau\psi^{(i)})\,dx\,, \end{aligned} \tag{2.36}$$

and solve the problem

$$\Delta\psi^{(i)} = (\rho_i^{\alpha_0,\alpha,\sigma})^{m_k}\ \tau\,.$$

In order to give sense to the terms with $\operatorname{div} u_{\alpha_0,\alpha,\sigma}^{(i)}$ and $\nabla\psi^{(i)}$ in (2.36), we have to choose m_k this time in such a way that

$$(\rho_i^{\alpha_0,\alpha,\sigma})^{m_k} \in L_{loc}^{1}(\Omega)\,.$$

Then, by the usual duality argument, it follows $\nabla\psi^{(i)} \in L_{loc}^{3/2-\delta}(\Omega;\mathbb{R}^3)$ for a small $\delta > 0$. This is more than sufficient since we can use $\nabla u_{\alpha_0,\alpha,\sigma}^{(i)} \in L_{loc}^{6}(\Omega;\mathbb{R}^{3\times 3})$ according to Part I (and thus $u_{\alpha_0,\alpha,\sigma}^{(i)} \in L_{loc}^{\infty}(\Omega;\mathbb{R}^3)$ due to the embedding; $u_{\alpha_0,\alpha,\sigma}^{(i)}$ is even Hölder continuous). To give sense to the integrals in equation (2.36) containing $\nabla\psi^{(i)}$, it would be sufficient to have $\nabla\psi^{(i)} \in L_{loc}^{6/5}(\Omega;\mathbb{R}^3)$.

According to the estimates proven in the first part of the proof, we know that $\rho_i^{\alpha_0,\alpha,\sigma} \in L_{loc}^{6}(\Omega)$ and we can choose now $m_0 = 6$. By the same procedure as in Part I we estimate $\rho_i^{\alpha_0,\alpha,\sigma}$ in $L_{loc}^{m_0+1}(\Omega) = L_{loc}^{7}(\Omega)$, and we can choose in the next step of the bootstrap argument $m_1 = 7$, and so on. In general, the sequence $m_k, k = 0,1,2,\dots,$ is defined by

$$m_0 = 6\,, \qquad m_{k+1} = m_k + 1\,,$$

such that we can expect finally estimates for $\rho_i^{\alpha_0,\alpha,\sigma}$ and thus also for $\nabla u_{\alpha_0,\alpha,\sigma}^{(i)}$ in all L_{loc}^p, $1 \le p < \infty$.

We consider equation (2.36) to obtain the estimate for $\rho_i^{\alpha_0,\alpha,\sigma} \in L_{loc}^{m_k+1}(\Omega)$.

The structure of the proof is the same as in Part I with the difference that we can now use $\nabla u^{(i)}_{\alpha_0,\alpha,\sigma} \in L^6_{loc}(\Omega;\mathbb{R}^{3\times 3})$ and $\rho_i^{\alpha_0,\alpha,\sigma} \in L^6_{loc}(\Omega)$ instead of only $\nabla u^{(i)}_{\alpha_0,\alpha,\sigma} \in L^2(\Omega;\mathbb{R}^{3\times 3})$ and $\rho_i^{\alpha_0,\alpha,\sigma} \in L^2(\Omega)$.

The term with $\operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma}$ in (2.36) gives

$$\int_\Omega \hat{\beta}_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \Delta\left(\tau\psi^{(i)}\right) dx = \int_\Omega \hat{\beta}_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \Delta\tau\psi^{(i)}\, dx$$
$$+2\int_\Omega \hat{\beta}_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \nabla\tau\cdot\nabla\psi^{(i)}\, dx + \int_\Omega \hat{\beta}_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \tau\Delta\psi^{(i)}\, dx\,,$$

where the first two integrals are bounded due to $\nabla u^{(i)}_{\alpha_0,\alpha,\sigma} \in L^6_{loc}(\Omega;\mathbb{R}^{3\times 3})$ and $\nabla\psi^{(i)} \in L^{3/2-\delta}_{loc}(\Omega;\mathbb{R}^3)$ ($\nabla\psi^{(i)} \in L^{6/5}_{loc}(\Omega;\mathbb{R}^3)$ would be sufficient).

Like in Part I, the third integral gives the contribution

$$\int_\Omega \hat{\beta}_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma}\tau^2 \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_k} dx = -\int_\Omega \hat{\beta}_i u^{(i)}_{\alpha_0,\alpha,\sigma}\cdot\nabla\tau^2 \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_k} dx$$
$$+\int_\Omega \hat{\beta}_i \operatorname{div}\left(\rho_i^{\alpha_0,\alpha,\sigma} u^{(i)}_{\alpha_0,\alpha,\sigma}\right)\frac{m_k}{m_k-1}\left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_k-1}\tau^2\, dx$$
$$+\int_\Omega \hat{\beta}_i \frac{m_k}{m_k-1}\left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_k} u^{(i)}_{\alpha_0,\alpha,\sigma}\cdot\nabla\tau^2\, dx\,.$$

The first and the third term are bounded due to $u^{(i)}_{\alpha_0,\alpha,\sigma} \in L^\infty_{loc}(\Omega;\mathbb{R}^3)$ and $(\rho_i^{\alpha_0,\alpha,\sigma})^{m_k} \in L^1_{loc}(\Omega)$ (because of the choice of m_k). The second integral is treated using equation (2.13) as in the first part of the proof of this theorem. The resulting terms are bounded and go to zero as α_0, α and σ tend to zero.

The terms with f and $I_{\alpha_0,\alpha,\sigma}$ in (2.36) are bounded, too, and it remains

$$\int_\Omega \left(A_0 P(\rho^{\alpha_0,\alpha,\sigma})\right)_i \Delta\left(\tau\psi^{(i)}\right) dx \le K\,,$$

which gives

$$\int_\Omega \left(A_0 P(\rho^{\alpha_0,\alpha,\sigma})\right)_i \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{m_k}\tau^2\, dx \le K\,.$$

Summing over i and using the condition (2.33), we get

$$\int_\Omega \left|\rho^{\alpha_0,\alpha,\sigma}\right|^{m_k+1}\tau^2\, dx \le K\,,$$

where $m_k, k = 0,1,2,\ldots$, is chosen as $m_0 = 6, m_{k+1} = m_k + 1$. With this choice of m_k, we obtain finally that

$$\int_\Omega \left|\rho^{\alpha_0,\alpha,\sigma}\right|^p \tau^2\, dx \le K \text{ for all } p \text{ with } 1 \le p < \infty\,.$$

In other words, we have proved that $\rho_i^{\alpha_0,\alpha,\sigma} \in L^p_{loc}(\Omega)$ for all $p, 1 \leq p < \infty$, and it follows from equation (2.14) that

$$\nabla u^{(i)}_{\alpha_0,\alpha,\sigma} \in L^p_{loc}(\Omega;\mathbb{R}^{3\times 3}) \text{ for all } p\,, 1 \leq p < \infty\,,$$

with estimates which are uniform in α_0, α and σ. □

Since the estimates in Theorem 2.3 are uniform with respect to α_0, α and σ, they hold also for the limit functions ρ_i and $u^{(i)}$, and Theorem 2.2 follows. The property $\nabla u^{(i)} \in L^p_{loc}$ for all $1 \leq p < \infty$ implies in particular the Hölder continuity of the velocity fields.

Remark: *The proof works also for more general pressure laws which behave like* $|\rho|^\gamma, \gamma > 1$, *as long as a coerciveness condition analogous to inequality (2.33) is fulfilled. Thus, we can prove for solutions of the Stokes system in a bounded domain* $\Omega \subset \mathbb{R}^3$ *local estimates for* ρ_i *and* $\nabla u^{(i)}$ *in* L^p *for all* p *with* $1 \leq p < \infty$ *also for other pressure laws, but for these cases existence of weak solutions is up to now not known since we are not able to obtain an* H^1*-estimate for* u.

It would be very interesting to prove an L^∞-estimate for the densities. For the Stokes system this is an open problem. In Chapter 5 of this thesis we prove an exponential estimate for the densities for the Stokes problem with a pressure which behaves like $|\rho|^\gamma, \gamma > 2$. This has to be seen as a first step in this direction.

2.3 Compactness of the densities

Even though weak convergence of the densities is sufficient to pass to the limit in the approximate equations (2.13)–(2.14) with interaction terms of the form (2.10), we show now that the approximate densities converge even *strongly*. In particular, this is important to be able to treat cases with different interaction terms (2.11), where the factor a depends in a nonlinear fashion on ρ. In these cases we need the *strong* convergence of the approximate densities to pass to the limit in the nonlinearity to obtain the existence of weak solutions.

In the proof we will see that we need – at least locally – more regularity for the densities than just L^2, which we have proved in the existence part. Since we would like to use the regularity which we have shown in the preceding section, we have to impose the coerciveness condition for the pressure in this section as well.

In order to obtain the strong convergence of the densities, we need in particular the coerciveness condition for $m = 1$, which is the positive definiteness of the matrix $A_0 \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$.

We can ensure the positive definiteness of $A_0 \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$ by choosing the parameter β_0 appropriately, as shown in the previous section on regularity.

Theorem 2.4 *Let the pressure P satisfy the coerciveness condition (2.33). Let the matrix $A_0 \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$ be positive definite.*
Let $(\rho^{\alpha_0,\alpha,\sigma}, u_{\alpha_0,\alpha,\sigma})$, $\rho^{\alpha_0,\alpha,\sigma} = (\rho_1^{\alpha_0,\alpha,\sigma}, \rho_2^{\alpha_0,\alpha,\sigma})^T, u_{\alpha_0,\alpha,\sigma} = \left(u_{\alpha_0,\alpha,\sigma}^{(1),T}, u_{\alpha_0,\alpha,\sigma}^{(2),T}\right)^T$, be solutions of the approximate system (2.13)–(2.15) with

$$\rho_i^{\alpha_0,\alpha,\sigma} \rightharpoonup \rho_i \text{ weakly in } L^2 \,,$$

and

$$u_{\alpha_0,\alpha,\sigma}^{(i)} \rightharpoonup u^{(i)} \text{ weakly in } H_0^1 \,, \quad i = 1, 2 \,,$$

where $(\rho, u), \rho = (\rho_1, \rho_2)^T, u = (u^{(1),T}, u^{(2),T})^T$, solves (2.1)–(2.3), (2.5). Then

$$\rho_i^{\alpha_0,\alpha,\sigma} \rightarrow \rho_i \text{ strongly in } L^r \,, 1 \le r < 2 \,, \text{ as } \alpha_0 \rightarrow 0 \text{ and } \alpha, \sigma \rightarrow 0 \,.$$

Proof:
As in the previous section we consider again the equation for the effective viscous flux, which is the key tool in this proof.

In the mathematical theory for compressible flow for *one* constituent, the so-called *effective viscous flux* or *effective pressure*, which is given by

$$p(\rho) - (\lambda + 2\mu) \operatorname{div} u \,,$$

p denoting the pressure, plays a key role in proving the compactness of the densities.

Concerning the convergence of the approximate to the primary quantities, the effective viscous flux possesses better properties than its ingredients.

In fact, in the steady case (cf. [Nov96], [Nov98]) or in the case of small initial data (cf. [Hof95a], [Hof95b]), one can even prove the strong convergence of

$$p(\rho^m) - (\lambda + 2\mu) \operatorname{div} u^m$$

(for approximate solutions ρ^m, u^m) to

$$\overline{p(\rho)} - (\lambda + 2\mu) \operatorname{div} u \,,$$

where $\overline{p(\rho)}$ denotes the weak limit of $p(\rho^m)$ as $m \rightarrow \infty$.

In the general evolutionary case, the product

$$\beta(\rho^m)\,(p(\rho^m) - (\lambda + 2\mu) \operatorname{div} u^m)$$

with a suitably chosen function β converges weakly to

$$\overline{\beta(\rho)}\left(\overline{p(\rho)}-(\lambda+2\mu)\operatorname{div}u\right).$$

This result is for instance treated in [Lio98, Chapter 5] and [Lio98, Appendix B]. A different proof of this convergence theorem with the aid of the div-curl lemma can be found in [Fei01] and [FNP01].

We make also use of the equation fulfilled by the analogous quantity in the case of mixtures to show the strong convergence of the densities.

For $i=1,2$ we have, as derived in the previous section,

$$\begin{aligned}\int_\Omega \hat{\beta}_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma}\Delta(\tau\psi^{(i)})\,dx &= \int_\Omega \left(A_0 P(\rho^{\alpha_0,\alpha,\sigma})\right)_i \Delta(\tau\psi^{(i)})\,dx \\ &- \int_\Omega \left(A_0 \operatorname{div}\Delta^{-1}\left(\rho^{\alpha_0,\alpha,\sigma} f + I_{\alpha_0,\alpha,\sigma}\right)\right)_i \Delta(\tau\psi^{(i)})\,dx\,,\end{aligned} \tag{2.37}$$

whereby

$$\hat{\beta}_i = \begin{cases} \beta_0\,, & i=1\,,\\ 1\,, & i=2\,,\end{cases}$$

$$P(\rho^{\alpha_0,\alpha,\sigma}) = \begin{pmatrix} c_1\rho_1^{\alpha_0,\alpha,\sigma} \\ c_2\rho_2^{\alpha_0,\alpha,\sigma}\end{pmatrix} \text{ and}$$

$$\operatorname{div}\Delta^{-1}\left(\rho^{\alpha_0,\alpha,\sigma} f + I_{\alpha_0,\alpha,\sigma}\right) = \begin{pmatrix} \operatorname{div}\Delta^{-1}\left(\rho_1^{\alpha_0,\alpha,\sigma} f^{(1)} + I^{(1)}_{\alpha_0,\alpha,\sigma}\right) \\ \operatorname{div}\Delta^{-1}\left(\rho_2^{\alpha_0,\alpha,\sigma} f^{(2)} + I^{(2)}_{\alpha_0,\alpha,\sigma}\right)\end{pmatrix}.$$

Due to the estimates from the first section of this chapter we know that for a subsequence

$$\begin{aligned} \rho_i^{\alpha_0,\alpha,\sigma} &\rightharpoonup \rho_i \text{ weakly in } L^2\,,\\ u^{(i)}_{\alpha_0,\alpha,\sigma} &\rightharpoonup u^{(i)} \text{ weakly in } H^1_0 \text{ and, due to the compact embedding,}\\ u^{(i)}_{\alpha_0,\alpha,\sigma} &\to u^{(i)} \text{ strongly in } L^q, q\in[1,6)\end{aligned}$$

as $\alpha_0 \to 0$ and then $\alpha,\sigma\to 0$. This was already sufficient to pass to the limit in the equations. We will show now that the approximate densities $\rho_i^{\alpha_0,\alpha,\sigma}$ converge even *strongly* in $L^r, 1\le r<2$, to ρ_i as $\alpha_0,\alpha,\sigma\to 0$.

We choose $\psi^{(i)}$ now by solving the problem

$$\Delta\psi^{(i)} = (\rho_i^{\alpha_0,\alpha,\sigma}-\rho_i)\,\tau \text{ in } \mathbb{R}^3\,.$$

Here, the functions $\rho_i^{\alpha_0,\alpha,\sigma}$ and ρ_i are extended by zero outside of Ω, τ is a smooth localization function in Ω which is assumed to be 1 inside of Ω and zero near the boundary of the domain.

As mentioned in the proof of regularity, the extended quantities $\rho_i^{\alpha_0,\alpha,\sigma}$ and $u^{(i)}_{\alpha_0,\alpha,\sigma}$ fulfill again the approximate continuity equation (2.13) in Ω and due to the boundary conditions they obey to

$$\operatorname{div}\left(\rho_i^{\alpha_0,\alpha,\sigma} u^{(i)}_{\alpha_0,\alpha,\sigma}\right) = 0 \text{ in } \mathbb{R}^3 \setminus \Omega\,.$$

Analogously, the functions ρ_i and $u^{(i)}$ which are extended by zero outside Ω satisfy again the continuity equation (2.1) as stated in Lemma 2.1 in [NN02]:

Lemma *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $\rho \in L^p(\Omega), p \geq 2, u \in H_0^1(\Omega;\mathbb{R}^3)$ and $f \in L^1(\Omega)$. Assume that*

$$\operatorname{div}(\rho u) = f \ \text{ in } \mathcal{D}'(\Omega)\,.$$

Then, extending ρ, u and f by zero outside Ω and denoting the new functions again by ρ, u and f, we have

$$\operatorname{div}(\rho u) = f \ \text{ in } \mathcal{D}'(\mathbb{R}^3)\,.$$

With the above choice of $\psi^{(i)}$ equation (2.37) becomes for $i = 1, 2$

$$\begin{aligned}
&\int_\Omega \hat\beta_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \Delta\left(\tau\Delta^{-1}\left(\left(\rho_i^{\alpha_0,\alpha,\sigma} - \rho_i\right)\tau\right)\right) dx \\
&= \int_\Omega \left(A_0 P(\rho^{\alpha_0,\alpha,\sigma})\right)_i \Delta\left(\tau\Delta^{-1}\left(\left(\rho_i^{\alpha_0,\alpha,\sigma} - \rho_i\right)\tau\right)\right) dx \\
&\quad - \int_\Omega \left(A_0 \operatorname{div}\Delta^{-1}\left(\rho^{\alpha_0,\alpha,\sigma} f + I_{\alpha_0,\alpha,\sigma}\right)\right)_i \Delta\left(\tau\Delta^{-1}\left(\left(\rho_i^{\alpha_0,\alpha,\sigma} - \rho_i\right)\tau\right)\right) dx\,.
\end{aligned} \tag{2.38}$$

We consider first the left-hand side of this equation and perform the limit process as α_0 and then α and σ tend to 0.

$$\begin{aligned}
&\int_\Omega \hat\beta_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \Delta\left(\tau\Delta^{-1}\left(\left(\rho_i^{\alpha_0,\alpha,\sigma} - \rho_i\right)\tau\right)\right) dx \\
&\quad = \int_\Omega \hat\beta_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \left(\rho_i^{\alpha_0,\alpha,\sigma} - \rho_i\right)\tau^2\, dx \\
&\qquad +2\int_\Omega \hat\beta_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \nabla\tau \cdot \nabla\Delta^{-1}\left(\left(\rho_i^{\alpha_0,\alpha,\sigma} - \rho_i\right)\tau\right) dx \\
&\qquad + \int_\Omega \hat\beta_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \Delta\tau\Delta^{-1}\left(\left(\rho_i^{\alpha_0,\alpha,\sigma} - \rho_i\right)\tau\right) dx \\
&= \int_\Omega \hat\beta_i \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \left(\rho_i^{\alpha_0,\alpha,\sigma} - \rho_i\right)\tau^2\, dx + o(1) \text{ as } \alpha_0 \to 0 \text{ and } \alpha, \sigma \to 0
\end{aligned}$$

due to the L^p-inclusions of the terms with $\nabla\Delta^{-1}$ and Δ^{-1}.

Consider for $i = 1, 2$

$$\begin{aligned}&\int_\Omega \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \left(\rho_i^{\alpha_0,\alpha,\sigma} - \rho_i\right) \tau^2 \, dx \\ &= \int_\Omega \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \rho_i^{\alpha_0,\alpha,\sigma} \tau^2 \, dx - \int_\Omega \operatorname{div} u^{(i)}_{\alpha_0,\alpha,\sigma} \rho_i \tau^2 \, dx \\ &= A_1^{(i)} + A_2^{(i)} \, .\end{aligned}$$

For reasons of lucidity, we omit additional indices referring to α_0, α, σ although the terms $A_1^{(i)}, A_2^{(i)}$ depend of course on these parameters.

The first integral is treated as follows with the aid of integration by parts

$$\begin{aligned}A_1^{(i)} &= -\int_\Omega u^{(i)}_{\alpha_0,\alpha,\sigma} \cdot \nabla \rho_i^{\alpha_0,\alpha,\sigma} \tau^2 \, dx - \int_\Omega \rho_i^{\alpha_0,\alpha,\sigma} u^{(i)}_{\alpha_0,\alpha,\sigma} \cdot \nabla \tau^2 \, dx \\ &= A_{11}^{(i)} + A_{12}^{(i)} \, .\end{aligned}$$

Since $\rho_i^{\alpha_0,\alpha,\sigma}$ converges weakly in L^2 and $u^{(i)}_{\alpha_0,\alpha,\sigma}$ strongly in $L^q, q \in [1, 6)$, the second integral converges to

$$A_{12}^{(i)} \to -\int_\Omega \rho_i u^{(i)} \cdot \nabla \tau^2 \, dx$$

as α_0, α, σ tend to zero. The limit will cancel with a corresponding term coming from the integral $A_2^{(i)}$ (see below).

The other term is treated using an auxiliary parameter $\delta > 0$ which will vanish before we perform the limit as α_0 tends to zero and α and σ tend to zero.

$$\begin{aligned}A_{11}^{(i)} &= -\int_\Omega \left(\rho_i^{\alpha_0,\alpha,\sigma} + \delta\right) u^{(i)}_{\alpha_0,\alpha,\sigma} \cdot \nabla \log\left(\rho_i^{\alpha_0,\alpha,\sigma} + \delta\right) \tau^2 \, dx \\ &= \int_\Omega \operatorname{div}\left(\left(\rho_i^{\alpha_0,\alpha,\sigma} + \delta\right) u^{(i)}_{\alpha_0,\alpha,\sigma}\right) \log\left(\rho_i^{\alpha_0,\alpha,\sigma} + \delta\right) \tau^2 \, dx \\ &\quad + \int_\Omega \left(\rho_i^{\alpha_0,\alpha,\sigma} + \delta\right) \log\left(\rho_i^{\alpha_0,\alpha,\sigma} + \delta\right) u^{(i)}_{\alpha_0,\alpha,\sigma} \cdot \nabla \tau^2 \, dx = A_{111}^{(i)} + A_{112}^{(i)} \, .\end{aligned}$$

The first term is split into two parts. In the first part we can use the continuity equation (2.13), the second part tends to zero as δ tends to zero:

$$
\begin{aligned}
A_{111}^{(i)} &= \int_\Omega \operatorname{div}\left(\rho_i^{\alpha_0,\alpha,\sigma} u_{\alpha_0,\alpha,\sigma}^{(i)}\right) \log\left(\rho_i^{\alpha_0,\alpha,\sigma}+\delta\right)\tau^2\,dx \\
&\quad +\delta \int_\Omega \operatorname{div} u_{\alpha_0,\alpha,\sigma}^{(i)} \log\left(\rho_i^{\alpha_0,\alpha,\sigma}+\delta\right)\tau^2\,dx \\
&= \sigma \int_\Omega \Delta\rho_i^{\alpha_0,\alpha,\sigma} \log\left(\rho_i^{\alpha_0,\alpha,\sigma}+\delta\right)\tau^2\,dx - \alpha_0 \int_\Omega \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^s \log\left(\rho_i^{\alpha_0,\alpha,\sigma}+\delta\right)\tau^2\,dx \\
&\quad -\alpha \int_\Omega \rho_i^{\alpha_0,\alpha,\sigma} \log\left(\rho_i^{\alpha_0,\alpha,\sigma}+\delta\right)\tau^2\,dx + \frac{\alpha}{|\Omega|}\int_\Omega \log\left(\rho_i^{\alpha_0,\alpha,\sigma}+\delta\right)\tau^2\,dx \\
&\quad +\delta \int_\Omega \operatorname{div} u_{\alpha_0,\alpha,\sigma}^{(i)} \log\left(\rho_i^{\alpha_0,\alpha,\sigma}+\delta\right)\tau^2\,dx\,,
\end{aligned}
$$

where $\delta\int_\Omega \operatorname{div} u_{\alpha_0,\alpha,\sigma}^{(i)} \log\left(\rho_i^{\alpha_0,\alpha,\sigma}+\delta\right)\tau^2\,dx \to 0$ as $\delta\to 0$. The other terms converge to the corresponding integrals without δ as δ tends to 0.

Since we have from the continuity equation (2.13) the estimate

$$
\alpha_0 \int_\Omega \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^{s+1}\,dx + \alpha \int_\Omega \left(\rho_i^{\alpha_0,\alpha,\sigma}\right)^2\,dx \le K\,,
$$

the terms with α_0 and α will vanish for α_0 and α tending to zero.

Analyze now the term with σ using integration by parts and Young's inequality:

$$
\begin{aligned}
&\sigma \int_\Omega \Delta\rho_i^{\alpha_0,\alpha,\sigma} \log\left(\rho_i^{\alpha_0,\alpha,\sigma}+\delta\right)\tau^2\,dx \\
&= -\sigma \int_\Omega \nabla\rho_i^{\alpha_0,\alpha,\sigma}\cdot\nabla \log\left(\rho_i^{\alpha_0,\alpha,\sigma}+\delta\right)\tau^2\,dx \\
&\quad -\sigma \int_\Omega \nabla\rho_i^{\alpha_0,\alpha,\sigma}\cdot\nabla\tau 2\tau \log\left(\rho_i^{\alpha_0,\alpha,\sigma}+\delta\right)\,dx \\
&\le -\sigma \int_\Omega \frac{|\nabla\rho_i^{\alpha_0,\alpha,\sigma}|^2}{\rho_i^{\alpha_0,\alpha,\sigma}+\delta}\tau^2\,dx + \sigma \int_\Omega \frac{|\nabla\rho_i^{\alpha_0,\alpha,\sigma}|^2}{\rho_i^{\alpha_0,\alpha,\sigma}+\delta}\tau^2\,dx \\
&\quad +\sigma \int_\Omega \left(\rho_i^{\alpha_0,\alpha,\sigma}+\delta\right)|\log\left(\rho_i^{\alpha_0,\alpha,\sigma}+\delta\right)|^2|\nabla\tau|^2\,dx\,.
\end{aligned}
$$

The first two integrals cancel, the last one vanishes for $\sigma\to 0$ since we have the bound

$$
\sigma\int_\Omega |\nabla\rho_i^{\alpha_0,\alpha,\sigma}|^2\,dx \le K\,,\ \text{thus, } \sigma\int_\Omega |\rho_i^{\alpha_0,\alpha,\sigma}|^6\,dx \le K\,.
$$

To summarize, the term $A_{111}^{(i)}$ tends to zero as α_0, α and σ tend to zero.

We investigate now the limit process for the term $A_{112}^{(i)}$.

$$
\begin{aligned}
A_{112}^{(i)} &= \int_\Omega (\rho_i^{\alpha_0,\alpha,\sigma}+\delta)\log(\rho_i^{\alpha_0,\alpha,\sigma}+\delta)\, u_{\alpha_0,\alpha,\sigma}^{(i)}\cdot\nabla\tau^2\,dx \\
&\overset{\delta\to 0}{\longrightarrow} \int_\Omega \rho_i^{\alpha_0,\alpha,\sigma}\log\rho_i^{\alpha_0,\alpha,\sigma} u_{\alpha_0,\alpha,\sigma}^{(i)}\cdot\nabla\tau^2\,dx \\
&\overset{\alpha_0,\alpha,\sigma\to 0}{\longrightarrow} \int_\Omega \overline{\rho_i\log\rho_i}\, u^{(i)}\cdot\nabla\tau^2\,dx\,,
\end{aligned}
$$

where $\overline{\rho_i\log\rho_i}$ denotes the weak limit of $\rho_i^{\alpha_0,\alpha,\sigma}\log\rho_i^{\alpha_0,\alpha,\sigma}$ as $\alpha_0,\alpha,\sigma\to 0$.

Considering the localization function τ, we know that for a test function φ

$$\int_\Omega \varphi u^{(i)}\cdot\nabla\tau^2\,dx \to \int_{\partial\Omega}\varphi u^{(i)}\cdot\vec{n}\,dS$$

as $\tau\to 1$, $\vec{n}$ denoting the outer normal vector.

This follows from the fact that the difference of the above integrals can be estimated with the help of a Poincaré-type argument by $K_\varphi\int|\nabla u^{(i)}|^2\,dx$, where the integration takes place over a strip near the boundary of the domain Ω.

Due to the boundary conditions, $u^{(i)}\cdot\vec{n}$ vanishes on $\partial\Omega$, and $A_{112}^{(i)}$ vanishes as well in the limit.

We consider now the term $A_2^{(i)}$, which we treat with the aid of a lemma due to DiPerna and Lions (cf. [DL89]).

As α_0,α and σ tend to 0, the term $A_2^{(i)}$ converges to

$$A_2^{(i)} = -\int_\Omega \operatorname{div} u_{\alpha_0,\alpha,\sigma}\rho_i\tau^2\,dx \to -\int_\Omega \operatorname{div} u^{(i)}\rho_i\tau^2\,dx\,.$$

From this term we obtain by regularizing ρ_i by the convolution with a mollifier ω_h

$$
\begin{aligned}
& -\int_\Omega \operatorname{div} u^{(i)}\rho_i\tau^2\,dx \\
=\;& -\int_\Omega (\rho_i*\omega_h)\operatorname{div} u^{(i)}\tau^2\,dx + \varepsilon_h \text{ with } \varepsilon_h\to 0 \text{ as } h\to 0 \\
=\;& \int_\Omega \nabla(\rho_i*\omega_h)\cdot u^{(i)}\tau^2\,dx + \int_\Omega (\rho_i*\omega_h)u^{(i)}\cdot\nabla\tau^2\,dx + \varepsilon_h \\
=\;& \int_\Omega (\rho_i*\omega_h+\delta)u^{(i)}\cdot\nabla\log(\rho_i*\omega_h+\delta)\tau^2\,dx + \int_\Omega (\rho_i*\omega_h)u^{(i)}\cdot\nabla\tau^2\,dx + \varepsilon_h \\
=\;& A_{21}^{(i)} + A_{22}^{(i)} + \varepsilon_h
\end{aligned}
$$

with an auxiliary parameter $\delta > 0$, which will tend to zero before α_0 and then α and σ tend to zero.
Here,

$$A_{22}^{(i)} \to \int_\Omega \rho_i u^{(i)} \cdot \nabla \tau^2 \, dx = - \lim_{\alpha_0, \alpha, \sigma \to 0} A_{12}^{(i)} \text{ as } h \to 0 \,,$$

i.e. these terms cancel in the limit. We investigate now the term $A_{21}^{(i)}$.

$$\begin{aligned} A_{21}^{(i)} &= -\int_\Omega \operatorname{div}[(\rho_i * \omega_h + \delta) u^{(i)}] \log(\rho_i * \omega_h + \delta) \tau^2 \, dx \\ &\quad - \int_\Omega (\rho_i * \omega_h + \delta) \log(\rho_i * \omega_h + \delta) u^{(i)} \cdot \nabla \tau^2 \, dx \\ &= A_{211}^{(i)} + A_{212}^{(i)} \,. \end{aligned}$$

The second term gives

$$\begin{aligned} A_{212}^{(i)} &\overset{h \to 0}{\to} - \int_\Omega (\rho_i + \delta) \log(\rho_i + \delta) u^{(i)} \cdot \nabla \tau^2 \, dx \\ &\overset{\delta \to 0}{\to} - \int_\Omega \rho_i \log \rho_i u^{(i)} \cdot \nabla \tau^2 \, dx \,. \end{aligned}$$

As above, as the localization function τ tends to one, we have for a test function φ that

$$\int_\Omega \varphi u^{(i)} \cdot \nabla \tau^2 \, dx \to \int_{\partial\Omega} \varphi u^{(i)} \cdot \vec{n} \, dS \,.$$

Due to the boundary conditions which we have imposed, the integral vanishes in the limit as $\tau \to 1$.

The term $A_{211}^{(i)}$ gives

$$\begin{aligned} A_{211}^{(i)} &= - \int_\Omega \operatorname{div} \left[(\rho_i * \omega_h) \, u^{(i)} \right] \log \left(\rho_i * \omega_h + \delta \right) \tau^2 \, dx \\ &\quad - \delta \int_\Omega \operatorname{div} u^{(i)} \log \left(\rho_i * \omega_h + \delta \right) \tau^2 \, dx \\ &= A_{2111}^{(i)} + A_{2112}^{(i)} \,. \end{aligned}$$

Here, the second part vanishes as $\delta \to 0$:

$$\begin{aligned} A_{2112}^{(i)} &\overset{h \to 0}{\to} -\delta \int_\Omega \operatorname{div} u^{(i)} \log(\rho_i + \delta) \tau^2 \, dx \\ &\overset{\delta \to 0}{\to} 0 \,. \end{aligned}$$

We will now treat the term $A_{2111}^{(i)}$ with a lemma by DiPerna and Lions using that $\operatorname{div}(\rho_i u^{(i)}) = 0$ weakly. Due to the global estimates from the first section, $\rho_i \in$

$L^2(\Omega), \nabla u^{(i)} \in L^2(\Omega;\mathbb{R}^{3\times 3})$, we can conclude by the DiPerna–Lions Lemma, which is a generalization of Friedrichs' lemma about commutators, that

$$\tau^2\left(\mathrm{div}[(\rho_i * \omega_h)u^{(i)}] - \mathrm{div}[\omega_h * (\rho_i u^{(i)})]\right) \rightharpoonup 0 \text{ weakly in } L^1 \text{ as } h \to 0\,.$$

This is not sufficient to pass to the limit for $h \to 0$ in the term

$$A^{(i)}_{2111} = -\int_\Omega \mathrm{div}[(\rho_i * \omega_h)u^{(i)}]\log(\rho_i * \omega_h + \delta)\tau^2\,dx\,,$$

where $\log(\rho_i * \omega_h + \delta) \in L^p(\Omega)$ for all $p, 1 \le p < \infty$, such that we have to use the local estimates on the density ρ_i from the previous section. These estimates ensure that

$$\tau^2\left(\mathrm{div}[(\rho_i * \omega_h)u^{(i)}] - \mathrm{div}[\omega_h * (\rho_i u^{(i)})]\right) \rightharpoonup 0 \text{ weakly in } L^2\,,\ \text{say, as } h \to 0\,,$$

and we can pass to the limit in the term under consideration. Since $\mathrm{div}(\rho_i u^{(i)}) = 0$, it follows that

$$A^{(i)}_{2111} \to 0 \text{ as } h \to 0\,.$$

Altogether, the left-hand side of (2.38) gives finally

$$\begin{aligned}&\int_\Omega \hat\beta_i\,\mathrm{div}\,u^{(i)}_{\alpha_0,\alpha,\sigma}\Delta(\tau\Delta^{-1}((\rho_i^{\alpha_0,\alpha,\sigma} - \rho_i)\tau))\,dx\\ &= \int_\Omega \hat\beta_i\,\mathrm{div}\,u^{(i)}_{\alpha_0,\alpha,\sigma}\,(\rho_i^{\alpha_0,\alpha,\sigma} - \rho_i)\,\tau^2\,dx + o(1) \to 0 \text{ as } \alpha_0 \to 0, \alpha, \sigma \to 0, \tau \to 1\,.\end{aligned}$$

Since the terms containing $f^{(i)}$ and $I^{(i)}_{\alpha_0,\alpha,\sigma}$ in (2.38), which are of lower order, tend to 0 as well, we have that

$$\begin{aligned}&\int_\Omega (A_0 P(\rho^{\alpha_0,\alpha,\sigma}))_i\,\Delta\left(\tau\Delta^{-1}\left((\rho_i^{\alpha_0,\alpha,\sigma} - \rho_i)\,\tau\right)\right)\,dx\\ &= \int_\Omega (A_0 P(\rho^{\alpha_0,\alpha,\sigma}))_i\,(\rho_i^{\alpha_0,\alpha,\sigma} - \rho_i)\,\tau^2\,dx + o(1) \to 0 \text{ as } \alpha_0 \to 0, \alpha, \sigma \to 0, \tau \to 1\,.\end{aligned}$$

From the convergence

$$\int_\Omega (\rho^{\alpha_0,\alpha,\sigma} - \rho)^T A_0 \begin{pmatrix} c_1\rho_1^{\alpha_0,\alpha,\sigma} \\ c_2\rho_2^{\alpha_0,\alpha,\sigma}\end{pmatrix}\tau^2\,dx \to 0 \text{ as } \alpha_0 \to 0, \alpha, \sigma \to 0, \tau \to 1 \qquad (2.39)$$

we can conclude that

$$\rho_i^{\alpha_0,\alpha,\sigma} \to \rho_i \text{ a.e. in } \Omega\,.$$

This is a consequence of the positive definiteness of the matrix $A_0\begin{pmatrix} c_1 & 0 \\ 0 & c_2\end{pmatrix}$.

In fact, we can conclude from the positive definiteness that for a $\lambda_0 > 0$ and for all $\rho, \hat{\rho}, \rho = (\rho_1, \rho_2)^T, \hat{\rho} = (\hat{\rho}_1, \hat{\rho}_2)^T$,

$$(\rho - \hat{\rho})^T A_0 \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} (\rho - \hat{\rho}) \geq \lambda_0 |\rho - \hat{\rho}|^2 \,,$$

such that the convergence of the $\rho_i^{\alpha_0,\alpha,\sigma}$ almost everywhere follows from (2.39).

Since we already know that $\|\rho_i^{\alpha_0,\alpha,\sigma}\|_{L^2} \leq K$ uniformly, the strong convergence

$$\rho_i^{\alpha_0,\alpha,\sigma} \longrightarrow \rho_i \text{ in } L^r \,, 1 \leq r < 2 \,, \text{ for } i = 1, 2$$

follows. □

Remark: *If we have the strong convergence of the densities, the strong convergence of* $\nabla u^{(i)}_{\alpha_0,\alpha,\sigma}$ *also follows:*

$$\nabla u^{(i)}_{\alpha_0,\alpha,\sigma} \longrightarrow \nabla u^{(i)} \textit{ strongly in } L^r \,, 1 \leq r < 2 \,, \textit{ as } \alpha_0 \to 0, \alpha, \sigma \to 0 \,.$$

Chapter 3

Compactness of solutions to the mixture model in the steady case

In this chapter we deal with the mixture model with convective terms in the steady case. We consider a sequence of solutions to the equations fulfilling certain bounds and show that the limit of this sequence is a solution of the equations as well.
This property – often called *compactness* (or weak sequential stability) – is considered to be the main step in an existence proof. In order to show the existence of solutions one has to construct approximate solutions which fulfill certain estimates. The compactness property then allows to pass to the limit in the equations.
In our model, however, it is not obvious how to obtain approximate solutions which fulfill appropriate a priori estimates.
Nevertheless, we would like to treat the compactness under the assumption of suitable estimates for ρ and u.

We consider the following set of equations for $i = 1, 2$, which are assumed to hold in a bounded open connected domain $\Omega \subset \mathbb{R}^3$:

$$\operatorname{div}\left(\rho_i u^{(i)}\right) = 0\,, \tag{3.1}$$

$$\sum_{k=1}^{2} L_{ik} u^{(k)} + \operatorname{div}\left(\rho_i u^{(i)} \otimes u^{(i)}\right) = -\nabla P_i(\rho) + \rho_i f^{(i)} + I^{(i)}\,, \tag{3.2}$$

$$\rho_i \geq 0\,, \quad \int_\Omega \rho_i \, dx = M > 0 \text{ given, say, } M = 1\,, \tag{3.3}$$

where the operators L_{ik} are given by

$$L_{ik} = -\mu_{ik}\Delta - (\lambda_{ik} + \mu_{ik})\nabla \operatorname{div}$$

fulfilling the ellipticity condition

$$\sum_{i,k=1}^{2} \int_\Omega L_{ik} u^{(k)} \cdot u^{(i)} \, dx \geq c_0 \int_\Omega |\nabla u|^2 \, dx \tag{3.4}$$

for a constant $c_0 > 0$.

The pressure $P(\rho) = (P_1(\rho), P_2(\rho))^T$ is assumed to satisfy the following conditions: We suppose that there are $\gamma > 1$ and $\beta_0 \neq 0$ such that $P(\rho)$ fulfills

(i) *the monotonicity condition:*
there is $\lambda_0 > 0$ such that for all $\rho = (\rho_1, \rho_2)^T$ and $\hat{\rho} = (\hat{\rho}_1, \hat{\rho}_2)^T, \rho_i, \hat{\rho}_i \geq 0$,

$$(\rho - \hat{\rho})^T A_0 \left(P(\rho) - P(\hat{\rho})\right) \geq \lambda_0 \left(|\rho|^{\gamma-1} + |\hat{\rho}|^{\gamma-1}\right) |\rho - \hat{\rho}|^2 \,; \tag{3.5}$$

(ii) *the growth condition:*
there is $K_1 > 0$ such that for all $\rho = (\rho_1, \rho_2)^T$ and $\hat{\rho} = (\hat{\rho}_1, \hat{\rho}_2)^T$

$$|P(\rho) - P(\hat{\rho})| \leq K_1 \left(|\rho|^{\gamma-1} + |\hat{\rho}|^{\gamma-1}\right) |\rho - \hat{\rho}| \,. \tag{3.6}$$

We recall the definition of the matrix A_0:

$$A_0 := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} := \begin{pmatrix} \beta_0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2\mu_{11} + \lambda_{11} & 2\mu_{12} + \lambda_{12} \\ 2\mu_{21} + \lambda_{21} & 2\mu_{22} + \lambda_{22} \end{pmatrix}^{-1} .$$

The interaction terms are assumed to be of the form

$$I^{(i)} = (-1)^{i+1} a \left(u^{(2)} - u^{(1)}\right) \tag{3.7}$$

with $a > 0$ constant.

The equations (3.1)–(3.3) are complemented by no-slip boundary conditions for the velocities:

$$u^{(i)}\big|_{\partial\Omega} = 0 \,. \tag{3.8}$$

Remark: *The result of this chapter can be proved analogously also in the case of slip boundary conditions:*

$$u^{(i)} \cdot \vec{n}\big|_{\partial\Omega} = 0$$

complemented with natural boundary conditions, which are stated in the previous chapter, $\vec{n}$ denoting the outer normal vector.

We prove the following

Theorem 3.1 *Let Ω be a Lipschitz domain. Assume that $f_m^{(i)} \in L^\infty(\Omega; \mathbb{R}^3), i = 1, 2,$ such that $f_m^{(i)} \rightharpoonup f^{(i)}$ weak-* in L^∞ with $f^{(i)} \in L^\infty(\Omega; \mathbb{R}^3)$.*
Let $(\rho^m, u_m), \rho^m = (\rho_1^m, \rho_2^m)^T, u_m = \left(u_m^{(1),T}, u_m^{(2),T}\right)^T$, $m = 1, 2, \ldots,$ be weak solutions of the system (3.1)–(3.3), (3.8) with $f^{(i)}$ replaced by $f_m^{(i)}$ with L_{ik} satisfying the

ellipticity condition (3.4), the pressure satisfying the conditions (3.5)–(3.6) and the interaction terms being of the form (3.7).
The solutions (ρ^m, u_m) *are assumed to fulfill*

$$\|\rho^m\|_{L^p} \leq K \text{ and } \|u_m\|_{H_0^1} \leq K \tag{3.9}$$

with $p > \gamma, p > 3$ *and* K *independent of* m*. Assume further that as* $m \to \infty$

$$\begin{aligned} \rho_i^m &\rightharpoonup \rho_i \text{ weakly in } L^p\,, \\ u_m^{(i)} &\rightharpoonup u^{(i)} \text{ weakly in } H_0^1\,. \end{aligned}$$

Then the limit $(\rho, u), \rho = (\rho_1, \rho_2)^T, u = \left(u^{(1),T}, u^{(2),T}\right)^T$, *is a weak solution of the system (3.1)–(3.3), (3.8).*

Remark: *The condition that* $\rho_i^m \in L^p(\Omega)$ *with* $p > 3$ *seems to be rather restrictive, but we will see below that this choice corresponds to the case of* γ *being greater than 2. In fact, if we have* $\rho_i^m \in L^\gamma(\Omega), \gamma > 2, u_m^{(i)} \in H_0^1(\Omega; \mathbb{R}^3)$*, we can conclude that* $\rho_i^m \in L^p(\Omega)$ *for a* $p > 3$*.*

Proof:
Due to the assumptions, as $m \to \infty$,

$$\begin{aligned} \rho_i^m &\rightharpoonup \rho_i \text{ weakly in } L^p\,, \\ u_m^{(i)} &\rightharpoonup u^{(i)} \text{ weakly in } H_0^1\,, \text{ and, owing to the compact embedding,} \\ u_m^{(i)} &\to u^{(i)} \text{ strongly in } L^q\,, q \in [1,6) \end{aligned}$$

for $i = 1, 2$.

Thus, we can pass to the limit in the equations (3.1)–(3.3) fulfilled by (ρ^m, u_m) in the weak sense and obtain that the limit (ρ, u), $\rho_i \in L^p(\Omega), u^{(i)} \in H_0^1(\Omega; \mathbb{R}^3)$, satisfies in the weak sense the following system of equations: For $i = 1, 2$

$$\operatorname{div}\left(\rho_i u^{(i)}\right) = 0\,, \tag{3.10}$$

$$\sum_{k=1}^{2} L_{ik} u^{(k)} + \operatorname{div}\left(\rho_i u^{(i)} \otimes u^{(i)}\right) = -\nabla \overline{P_i(\rho)} + \overline{\rho_i f^{(i)}} + I^{(i)}\,, \tag{3.11}$$

$$\rho_i \geq 0\,, \quad \int_\Omega \rho_i \, dx = 1\,, \tag{3.12}$$

where $\overline{P_i(\rho)}$ denotes the weak limit of $P_i(\rho^m)$ as $m \to \infty$ in $L^{p/\gamma}$, analogously $\overline{\rho_i f^{(i)}}$.

The main difficulty here is to show that

$$\overline{P_i(\rho)} = P_i(\rho) \tag{3.13}$$

for the nonlinear quantity $P_i(\rho)$, i.e. we have to prove the *strong* convergence of the densities:

$$\rho_i^m \to \rho_i \text{ strongly in } L^r\,, 1 \le r < p\,, \text{ as } m \to \infty\,.$$

Then, of course, also

$$\overline{\rho_i f^{(i)}} = \rho_i f^{(i)}\,.$$

For this purpose we make again use of our most important tool – the equation for the effective viscous flux. We adapt the techniques used in the theory of compressible flow to the case of mixtures.

We consider the weak formulation of equation (3.2) for (ρ^m, u_m) with a test function $\nabla\left(\tau\varphi^{(i)}\right)$, where $\tau \in \mathcal{D}(\Omega)$ is a localization function.

$$\int_\Omega \begin{pmatrix} \Delta\left(\tau\varphi^{(1)}\right) \\ \Delta\left(\tau\varphi^{(2)}\right) \end{pmatrix}^T \begin{pmatrix} 2\mu_{11}+\lambda_{11} & 2\mu_{12}+\lambda_{12} \\ 2\mu_{21}+\lambda_{21} & 2\mu_{22}+\lambda_{22} \end{pmatrix} \begin{pmatrix} \operatorname{div} u_m^{(1)} \\ \operatorname{div} u_m^{(2)} \end{pmatrix} dx$$

$$-\int_\Omega \begin{pmatrix} \Delta\left(\tau\varphi^{(1)}\right) \\ \Delta\left(\tau\varphi^{(2)}\right) \end{pmatrix}^T \begin{pmatrix} \operatorname{div}\Delta^{-1}\operatorname{div}\left(\rho_1^m u_m^{(1)} \otimes u_m^{(1)}\right) \\ \operatorname{div}\Delta^{-1}\operatorname{div}\left(\rho_2^m u_m^{(2)} \otimes u_m^{(2)}\right) \end{pmatrix} dx$$

$$= \int_\Omega \begin{pmatrix} \Delta\left(\tau\varphi^{(1)}\right) \\ \Delta\left(\tau\varphi^{(2)}\right) \end{pmatrix}^T \begin{pmatrix} P_1(\rho^m) \\ P_2(\rho^m) \end{pmatrix} dx$$

$$-\int_\Omega \begin{pmatrix} \Delta\left(\tau\varphi^{(1)}\right) \\ \Delta\left(\tau\varphi^{(2)}\right) \end{pmatrix}^T \begin{pmatrix} \operatorname{div}\Delta^{-1}\left(\rho_1^m f_m^{(1)} + I_m^{(1)}\right) \\ \operatorname{div}\Delta^{-1}\left(\rho_2^m f_m^{(2)} + I_m^{(2)}\right) \end{pmatrix} dx\,.$$

Here, we use the notation

$$I_m^{(i)} = (-1)^{i+1} a\left(u_m^{(2)} - u_m^{(1)}\right)\,.$$

The operator Δ^{-1} is understood as solving the Laplace equation in $\mathbb{R}^3$, and the functions under consideration are extended by zero outside the domain Ω.

We have used that for $i = 1, 2$

$$\int_\Omega \rho_i^m f_m^{(i)} \cdot \nabla\left(\tau\varphi^{(i)}\right) dx = -\int_\Omega \operatorname{div}\Delta^{-1}\left(\rho_i^m f_m^{(i)}\right) \Delta\left(\tau\varphi^{(i)}\right) dx$$

and

$$\int_\Omega I_m^{(i)} \cdot \nabla\left(\tau\varphi^{(i)}\right) dx = -\int_\Omega \operatorname{div}\Delta^{-1}\left(I_m^{(i)}\right) \Delta\left(\tau\varphi^{(i)}\right) dx$$

as in the previous chapter. Analogously,

$$\int_\Omega \operatorname{div}\left(\rho_i^m u_m^{(i)} \otimes u_m^{(i)}\right) \cdot \nabla\left(\tau\varphi^{(i)}\right) dx = -\int_\Omega \operatorname{div}\Delta^{-1}\operatorname{div}\left(\rho_i^m u_m^{(i)} \otimes u_m^{(i)}\right) \Delta\left(\tau\varphi^{(i)}\right) dx\,.$$

With

$$A_0 := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} := \begin{pmatrix} \beta_0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2\mu_{11}+\lambda_{11} & 2\mu_{12}+\lambda_{12} \\ 2\mu_{21}+\lambda_{21} & 2\mu_{22}+\lambda_{22} \end{pmatrix}^{-1}$$

and $\varphi = A_0^T \tilde{\psi}, \varphi = \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix}, \tilde{\psi} = \begin{pmatrix} \tilde{\psi}^{(1)} \\ \tilde{\psi}^{(2)} \end{pmatrix}$ this writes

$$\begin{aligned}
&\int_\Omega \begin{pmatrix} \Delta\left(\tau\tilde{\psi}^{(1)}\right) \\ \Delta\left(\tau\tilde{\psi}^{(2)}\right) \end{pmatrix}^T \begin{pmatrix} \beta_0 \operatorname{div} u_m^{(1)} \\ \operatorname{div} u_m^{(2)} \end{pmatrix} dx \\
&-\int_\Omega \begin{pmatrix} \Delta\left(\tau\tilde{\psi}^{(1)}\right) \\ \Delta\left(\tau\tilde{\psi}^{(2)}\right) \end{pmatrix}^T A_0 \begin{pmatrix} \operatorname{div}\Delta^{-1}\operatorname{div}\left(\rho_1^m u_m^{(1)} \otimes u_m^{(1)}\right) \\ \operatorname{div}\Delta^{-1}\operatorname{div}\left(\rho_2^m u_m^{(2)} \otimes u_m^{(2)}\right) \end{pmatrix} dx \\
&= \int_\Omega \begin{pmatrix} \Delta\left(\tau\tilde{\psi}^{(1)}\right) \\ \Delta\left(\tau\tilde{\psi}^{(2)}\right) \end{pmatrix}^T A_0 \begin{pmatrix} P_1(\rho^m) \\ P_2(\rho^m) \end{pmatrix} dx \\
&-\int_\Omega \begin{pmatrix} \Delta\left(\tau\tilde{\psi}^{(1)}\right) \\ \Delta\left(\tau\tilde{\psi}^{(2)}\right) \end{pmatrix}^T A_0 \begin{pmatrix} \operatorname{div}\Delta^{-1}\left(\rho_1^m f_m^{(1)} + I_m^{(1)}\right) \\ \operatorname{div}\Delta^{-1}\left(\rho_2^m f_m^{(2)} + I_m^{(2)}\right) \end{pmatrix} dx
\end{aligned}$$

By separating it follows for $i = 1, 2$

$$\begin{aligned}
&\int_\Omega \hat{\beta}_i \operatorname{div} u_m^{(i)} \Delta\left(\tau\psi^{(i)}\right) dx - \int_\Omega \left(A_0 P(\rho^m)\right)_i \Delta\left(\tau\psi^{(i)}\right) dx \\
&= \int_\Omega \left(A_0 \operatorname{div}\Delta^{-1}\operatorname{div}\left(\rho^m u_m \otimes u_m\right)\right)_i \Delta\left(\tau\psi^{(i)}\right) dx \\
&- \int_\Omega \left(A_0 \operatorname{div}\Delta^{-1}\left(\rho^m f_m + I_m\right)\right)_i \Delta\left(\tau\psi^{(i)}\right) dx \,,
\end{aligned} \tag{3.14}$$

where

$$\hat{\beta}_i = \begin{cases} \beta_0\,, & i = 1\,, \\ 1, & i = 2\,, \end{cases}$$

$$P(\rho^m) = \begin{pmatrix} P_1(\rho^m) \\ P_2(\rho^m) \end{pmatrix},$$

$$\operatorname{div}\Delta^{-1}\operatorname{div}\left(\rho^m u_m \otimes u_m\right) = \begin{pmatrix} \operatorname{div}\Delta^{-1}\operatorname{div}\left(\rho_1^m u_m^{(1)} \otimes u_m^{(1)}\right) \\ \operatorname{div}\Delta^{-1}\operatorname{div}\left(\rho_2^m u_m^{(2)} \otimes u_m^{(2)}\right) \end{pmatrix},$$

$$\operatorname{div}\Delta^{-1}\left(\rho^m f_m + I_m\right) = \begin{pmatrix} \operatorname{div}\Delta^{-1}\left(\rho_1^m f_m^{(1)} + I_m^{(1)}\right) \\ \operatorname{div}\Delta^{-1}\left(\rho_2^m f_m^{(2)} + I_m^{(2)}\right) \end{pmatrix}.$$

Thus, we have derived with equation (3.14) the equation for the effective viscous flux for the steady mixture model with convective terms.

Now, we choose a special function $\psi^{(i)}$. We solve the problem

$$\Delta\psi^{(i)} = (\rho_i^m - \rho_i)\,\tau$$

in $\mathbb{R}^3$, where $\tau \in \mathcal{D}(\Omega)$ is a localization function and ρ_i^m and ρ_i are extended by zero outside Ω.
The extended functions solve the continuity equation (3.1) in $\mathcal{D}'(\mathbb{R}^3)$, cf. Lemma 2.1 in [NN02].

With the above choice of $\psi^{(i)}$, we have to consider

$$\begin{aligned}
&\int_\Omega \hat{\beta}_i \operatorname{div} u_m^{(i)} \Delta\left(\tau\Delta^{-1}\left((\rho_i^m - \rho_i)\,\tau\right)\right)\,dx \\
&\quad - \int_\Omega \left(A_0 P(\rho^m)\right)_i \Delta\left(\tau\Delta^{-1}\left((\rho_i^m - \rho_i)\,\tau\right)\right)\,dx \\
&= \int_\Omega \left(A_0 \operatorname{div}\Delta^{-1}\operatorname{div}\left(\rho^m u_m \otimes u_m\right)\right)_i \Delta\left(\tau\Delta^{-1}\left((\rho_i^m - \rho_i)\,\tau\right)\right)\,dx \\
&\quad - \int_\Omega \left(A_0 \operatorname{div}\Delta^{-1}\left(\rho^m f_m + I_m\right)\right)_i \Delta\left(\tau\Delta^{-1}\left((\rho_i^m - \rho_i)\,\tau\right)\right)\,dx
\end{aligned} \tag{3.15}$$

for $i = 1, 2$.

We want to prove that, as m tends to ∞,

$$\int_\Omega \left(A_0 P(\rho^m)\right)_i \Delta\left(\tau\Delta^{-1}\left((\rho_i^m - \rho_i)\,\tau\right)\right)\,dx \to 0\,.$$

Thus, we have to show that all other integrals in (3.15) vanish as $m \to \infty$.

Firstly, we investigate the most difficult terms, i.e. those coming from the nonlinear convective terms $\operatorname{div}(\rho_i^m u_m^{(i)} \otimes u_m^{(i)})$.

In the theory of compressible flow for one component, the convective term (together with a term coming from the time derivative of ρu) is treated with the help of a commutator lemma (cf. [CLMS93]) in the book by Lions ([Lio98]) or by Feireisl et al. (cf. [Fei01], [FNP01]) with the div-curl lemma known from the theory of compensated compactness, which was founded by F. Murat and L. Tartar (cf. [Mur78], [Mur79], [Tar79]). For an L^p–L^q version of the div-curl lemma see also [Zho92]. We will make use of the div-curl lemma as well, but in the case of steady mixtures considered here the application is more direct since we obtain only from the convective term a quantity where we can apply the div-curl lemma directly, and we do not have to deal with $(\rho u)_t$ additionally.

We consider without loss of generality $i = 1$. The integral coming from the convective term writes with $A_0 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ as follows.

$$\begin{aligned}
&\int_\Omega \left(A_0 \operatorname{div} \Delta^{-1} \operatorname{div}\left(\rho^m u_m \otimes u_m\right)\right)_1 \Delta\left(\tau \Delta^{-1}\left(\left(\rho_1^m - \rho_1\right)\tau\right)\right) \, dx \\
&= \int_\Omega \left\{a_{11} \operatorname{div} \Delta^{-1} \operatorname{div}\left(\rho_1^m u_m^{(1)} \otimes u_m^{(1)}\right) + a_{12} \operatorname{div} \Delta^{-1} \operatorname{div}\left(\rho_2^m u_m^{(2)} \otimes u_m^{(2)}\right)\right\} \cdot \\
&\qquad \cdot \Delta\left(\tau \Delta^{-1}\left(\left(\rho_1^m - \rho_1\right)\tau\right)\right) \, dx \,,
\end{aligned}$$

which is in components (with summation convention, we sum over k and j from 1 to 3; the summation convention is exceptionally used here)

$$\begin{aligned}
&= \int_\Omega \left\{a_{11} \partial_j \Delta^{-1} \partial_k \left(\rho_1^m \left(u_m^{(1)}\right)_k \left(u_m^{(1)}\right)_j\right) + a_{12} \partial_j \Delta^{-1} \partial_k \left(\rho_2^m \left(u_m^{(2)}\right)_k \left(u_m^{(2)}\right)_j\right)\right\} \cdot \\
&\qquad \cdot \Delta\left(\tau \Delta^{-1}\left(\left(\rho_1^m - \rho_1\right)\tau\right)\right) \, dx \,.
\end{aligned}$$

Integration by parts leads to

$$\begin{aligned}
&= \int_\Omega \left\{a_{11} \rho_1^m \left(u_m^{(1)}\right)_k \left(u_m^{(1)}\right)_j + a_{12} \rho_2^m \left(u_m^{(2)}\right)_k \left(u_m^{(2)}\right)_j\right\} \cdot \\
&\qquad \cdot \partial_j \partial_k \left(\tau \Delta^{-1}\left(\left(\rho_1^m - \rho_1\right)\tau\right)\right) \, dx \\
&= \int_\Omega a_{11} \left(u_m^{(1)}\right)_k \rho_1^m \left(u_m^{(1)}\right)_j \partial_j \partial_k \left(\tau \Delta^{-1}\left(\left(\rho_1^m - \rho_1\right)\tau\right)\right) \, dx \\
&+ \int_\Omega a_{12} \left(u_m^{(2)}\right)_k \rho_2^m \left(u_m^{(2)}\right)_j \partial_j \partial_k \left(\tau \Delta^{-1}\left(\left(\rho_1^m - \rho_1\right)\tau\right)\right) \, dx \,.
\end{aligned}$$

Denoting $z_1^m := \partial_k \left(\tau \Delta^{-1}\left(\left(\rho_1^m - \rho_1\right)\tau\right)\right)$, we consider the terms for $i = 1, 2$

$$\rho_i^m \left(u_m^{(i)}\right)_j \partial_j z_1^m = \rho_i^m u_m^{(i)} \cdot \nabla z_1^m \,,$$

where the first part is divergence-free, $\operatorname{div}\left(\rho_i^m u_m^{(i)}\right) = 0$, and the second part is a gradient, and therefore $\operatorname{curl} \nabla z_1^m = 0$. With the aid of the div-curl lemma we can thus conclude that the product

$$\rho_i^m \left(u_m^{(i)}\right)_j \partial_j z_1^m \rightharpoonup \rho_i \left(u^{(i)}\right)_j \overline{\partial_j z_1} \quad \text{weakly in} \quad L^r \,,$$

where $\frac{1}{r} = \frac{1}{p} + \frac{6+p}{6p}$, i.e. $r = \frac{6p}{12+p}$. Here, $\overline{\partial_j z_1}$ denotes the weak limit of $\partial_j z_1^m$ in L^p. As $p > 3$, it is $r > \frac{6}{5}$.

Since

$$\rho_1^m - \rho_1 \rightharpoonup 0 \text{ as } m \to \infty \text{ in } L^p \,,$$

it is $\overline{\partial_j z_1} = 0$.
Since further

$$u_m^{(i)} \to u^{(i)} \text{ strongly in } L^q, q \in [1, 6) \text{ due to the compact embedding,}$$

the integrals

$$\int_\Omega a_{11}\left(u_m^{(1)}\right)_k \rho_1^m \left(u_m^{(1)}\right)_j \partial_j\partial_k \left(\tau\Delta^{-1}\left(\left(\rho_1^m-\rho_1\right)\tau\right)\right)\,dx$$
$$+\int_\Omega a_{12}\left(u_m^{(2)}\right)_k \rho_2^m \left(u_m^{(2)}\right)_j \partial_j\partial_k \left(\tau\Delta^{-1}\left(\left(\rho_1^m-\rho_1\right)\tau\right)\right)\,dx$$

vanish as $m\to\infty$.

Thus, as we pass to the limit in the equation (3.15) as $m\to\infty$, the terms coming from the convective terms tend to zero.

Let us remark that our approach is more direct than the one by Novo and Novotný ([NN02]) for the compressible Navier–Stokes equations in the stationary case. The method used in the paper [NN02] for obtaining the convergence of the integrals which result from the convective term imitates Feireisl's technique for the evolutionary case (cf. [Fei01]). However, it is not necessary to apply such a complicated tool in the steady case where one does not have to deal with the time derivative $(\rho u)_t$.

Next, we consider the terms coming from $\operatorname{div} u_m^{(i)}$. We obtain for $i=1,2$

$$\begin{aligned}
&\int_\Omega \hat\beta_i \operatorname{div} u_m^{(i)}\Delta\left(\tau\Delta^{-1}\left(\left(\rho_i^m-\rho_i\right)\tau\right)\right)\,dx\\
&=\int_\Omega \hat\beta_i \operatorname{div} u_m^{(i)}\Delta\tau\Delta^{-1}\left(\left(\rho_i^m-\rho_i\right)\tau\right)\,dx\\
&+2\int_\Omega \hat\beta_i \operatorname{div} u_m^{(i)}\nabla\tau\cdot\nabla\Delta^{-1}\left(\left(\rho_i^m-\rho_i\right)\tau\right)\,dx+\int_\Omega \hat\beta_i \operatorname{div} u_m^{(i)}\left(\rho_i^m-\rho_i\right)\tau^2\,dx\\
&=\int_\Omega \hat\beta_i \operatorname{div} u_m^{(i)}\left(\rho_i^m-\rho_i\right)\tau^2\,dx+o(1) \text{ as } m\to\infty
\end{aligned}$$

because of the better L^p-inclusions of the terms with Δ^{-1} and $\nabla\Delta^{-1}$ and $\rho_i^m \rightharpoonup \rho_i$ in L^p.

Investigating the remaining integral further gives for $i=1,2$

$$\begin{aligned}
&\int_\Omega \hat\beta_i \operatorname{div} u_m^{(i)}\left(\rho_i^m-\rho_i\right)\tau^2\,dx\\
&=\int_\Omega \hat\beta_i \operatorname{div} u_m^{(i)}\rho_i^m\tau^2\,dx-\int_\Omega \hat\beta_i \operatorname{div} u_m^{(i)}\rho_i\tau^2\,dx\\
&=B_{1m}^{(i)}+B_{2m}^{(i)}\,.
\end{aligned}$$

We treat the first integral by inserting a mollifier ω_h in order to regularize the density

ρ_i^m, which satisfies only a first-order equation and belongs only to a Lebesgue space.

$$\begin{aligned} B_{1m}^{(i)} &= \int_\Omega \hat{\beta}_i \left(\rho_i^m * \omega_h\right) \operatorname{div} u_m^{(i)} \tau^2 \, dx + \varepsilon_h, \text{ where } \varepsilon_h \to 0 \text{ as } h \to 0\,, \\ &= -\int_\Omega \hat{\beta}_i u_m^{(i)} \cdot \nabla \left(\rho_i^m * \omega_h\right) \tau^2 \, dx - \int_\Omega \hat{\beta}_i \left(\rho_i^m * \omega_h\right) u_m^{(i)} \cdot \nabla \tau^2 \, dx + \varepsilon_h \\ &= B_{11m}^{(i)} + B_{12m}^{(i)} + \varepsilon_h\,, \end{aligned}$$

where we used integration by parts.

As $h \to 0$ and afterwards $m \to \infty$, the second term converges to

$$\begin{aligned} B_{12m}^{(i)} &\overset{h\to 0}{\to} -\int_\Omega \hat{\beta}_i \rho_i^m u_m^{(i)} \cdot \nabla \tau^2 \, dx \\ &\overset{m\to\infty}{\to} -\int_\Omega \hat{\beta}_i \rho_i u^{(i)} \cdot \nabla \tau^2 \, dx\,, \end{aligned}$$

and this integral will cancel with the term $\lim_{h\to 0} B_{22}^{(i)}$ below coming from the part with $B_{2m}^{(i)}$.

We analyze now the term $B_{11m}^{(i)}$ more closely by inserting an auxiliary parameter $\delta > 0$, which will be sent to zero before m tends to infinity.

$$\begin{aligned} B_{11m}^{(i)} &= -\int_\Omega \hat{\beta}_i \left(\rho_i^m * \omega_h + \delta\right) u_m^{(i)} \cdot \nabla \log \left(\rho_i^m * \omega_h + \delta\right) \tau^2 \, dx \\ &= \int_\Omega \hat{\beta}_i \operatorname{div} \left[\left(\rho_i^m * \omega_h + \delta\right) u_m^{(i)}\right] \log \left(\rho_i^m * \omega_h + \delta\right) \tau^2 \, dx \\ &\quad + \int_\Omega \hat{\beta}_i \left(\rho_i^m * \omega_h + \delta\right) \log \left(\rho_i^m * \omega_h + \delta\right) u_m^{(i)} \cdot \nabla \tau^2 \, dx \\ &= B_{111m}^{(i)} + B_{112m}^{(i)}\,. \end{aligned}$$

The second integral has the following convergence properties

$$\begin{aligned} B_{112m}^{(i)} &\overset{h\to 0}{\to} \int_\Omega \hat{\beta}_i \left(\rho_i^m + \delta\right) \log \left(\rho_i^m + \delta\right) u_m^{(i)} \cdot \nabla \tau^2 \, dx \\ &\overset{\delta\to 0}{\to} \int_\Omega \hat{\beta}_i \rho_i^m \log \rho_i^m u_m^{(i)} \cdot \nabla \tau^2 \, dx \\ &\overset{m\to\infty}{\to} \int_\Omega \hat{\beta}_i \overline{\rho_i \log \rho_i} u^{(i)} \cdot \nabla \tau^2 \, dx\,, \end{aligned}$$

where $\overline{\rho_i \log \rho_i}$ denotes the weak limit of $\rho_i^m \log \rho_i^m$ as $m \to \infty$.
Due to the choice of the localization function τ we have as $\tau \to 1$ by using a Poincaré-type argument

$$\int_\Omega \varphi u^{(i)} \cdot \nabla \tau^2 \, dx \to \int_{\partial\Omega} \varphi u^{(i)} \cdot \vec{n} \, dS$$

for a test function φ, and the limit vanishes due to the boundary conditions (3.8).

The term $B^{(i)}_{111m}$ gives the following two terms

$$\begin{aligned} B^{(i)}_{111m} &= \int_\Omega \hat{\beta}_i \operatorname{div}\left[\left(\rho_i^m * \omega_h\right) u_m^{(i)}\right] \log\left(\rho_i^m * \omega_h + \delta\right) \tau^2 \, dx \\ &\quad + \delta \int_\Omega \hat{\beta}_i \operatorname{div} u_m^{(i)} \log\left(\rho_i^m * \omega_h + \delta\right) \tau^2 \, dx \\ &= B^{(i)}_{1111m} + B^{(i)}_{1112m} \, . \end{aligned}$$

The second integral vanishes as $\delta \to 0$:

$$\begin{aligned} B^{(i)}_{1112m} &\overset{h\to 0}{\to} \delta \int_\Omega \hat{\beta}_i \operatorname{div} u_m^{(i)} \log\left(\rho_i^m + \delta\right) \tau^2 \, dx \\ &\overset{\delta\to 0}{\to} 0 \, . \end{aligned}$$

The term $B^{(i)}_{1111m}$ is treated with the aid of a lemma by DiPerna and Lions ([DL89]). We want to pass to the limit as $h \to 0$ in this term and make use of the fact that $\operatorname{div}\left(\rho_i^m u_m^{(i)}\right) = 0$ weakly.
Due to the assumptions $\rho_i^m \in L^p(\Omega), p > 3, \nabla u_m^{(i)} \in L^2(\Omega; \mathbb{R}^{3\times 3})$ we can conclude by the DiPerna–Lions Lemma that

$$\tau^2 \left(\operatorname{div}\left[\left(\rho_i^m * \omega_h\right) u_m^{(i)}\right] - \operatorname{div}\left[\omega_h * \left(\rho_i^m u_m^{(i)}\right)\right]\right) \rightharpoonup 0 \text{ weakly in } L^{\frac{2p}{p+2}} \text{ as } h \to 0 \, .$$

Thus,

$$B^{(i)}_{1111m} = \int_\Omega \hat{\beta}_i \operatorname{div}\left[\left(\rho_i^m * \omega_h\right) u_m^{(i)}\right] \log\left(\rho_i^m * \omega_h + \delta\right) \tau^2 \, dx \to 0 \text{ as } h \to 0$$

since $\operatorname{div}\left(\rho_i^m u_m^{(i)}\right) = 0$ due to the assumptions of the theorem. We have used that $\log(\rho_i^m * \omega_h + \delta) \in L^q(\Omega)$ for all $q, 1 \le q < \infty$.

Now we have to analyze the term $B^{(i)}_{2m}$.
As $m \to \infty$, the integral converges to

$$B^{(i)}_{2m} \overset{m\to\infty}{\to} - \int_\Omega \hat{\beta}_i \operatorname{div} u^{(i)} \rho_i \tau^2 \, dx \, .$$

This term is treated like $B^{(i)}_{1m}$ by regularizing the limit density ρ_i by convoluting with a mollifier ω_h:

$$\lim_{m\to\infty} B^{(i)}_{2m} = - \int_\Omega \hat{\beta}_i \left(\rho_i * \omega_h\right) \operatorname{div} u^{(i)} \tau^2 \, dx + \varepsilon_h \text{ with } \varepsilon_h \to 0 \text{ as } h \to 0 \, .$$

The same manipulations as for the term $B^{(i)}_{1m}$ give

$$\begin{aligned}\lim_{m\to\infty} B^{(i)}_{2m} &= \int_\Omega \hat{\beta}_i u^{(i)} \cdot \nabla (\rho_i * \omega_h)\, \tau^2\, dx + \int_\Omega \hat{\beta}_i (\rho_i * \omega_h)\, u^{(i)} \cdot \nabla\tau^2\, dx + \varepsilon_h \\ &= B^{(i)}_{21} + B^{(i)}_{22} + \varepsilon_h\,,\end{aligned}$$

where $\lim_{h\to 0} B^{(i)}_{22}$ cancels with $\lim_{m\to\infty}\lim_{h\to 0} B^{(i)}_{12m}$ as mentioned above:

$$\lim_{h\to 0} B^{(i)}_{22} = \int_\Omega \hat{\beta}_i \rho_i u^{(i)} \cdot \nabla\tau^2\, dx = -\lim_{m\to\infty}\lim_{h\to 0} B^{(i)}_{12m}\,.$$

The term $B^{(i)}_{21}$ can be treated exactly like $B^{(i)}_{11m}$ above since the limit functions (ρ, u) satisfy the continuity equation as well as stated in (3.10).
This time, we have to consider only the limit processes with respect to the parameters h and δ. We obtain with a paramter $\delta > 0$

$$\begin{aligned}B^{(i)}_{21} &= \int_\Omega \hat{\beta}_i (\rho_i * \omega_h + \delta)\, u^{(i)} \cdot \nabla \log(\rho_i * \omega_h + \delta)\, \tau^2\, dx \\ &= -\int_\Omega \hat{\beta}_i \operatorname{div}\left[(\rho_i * \omega_h + \delta)\, u^{(i)}\right] \log(\rho_i * \omega_h + \delta)\, \tau^2\, dx \\ &\quad -\int_\Omega \hat{\beta}_i (\rho_i * \omega_h + \delta) \log(\rho_i * \omega_h + \delta)\, u^{(i)} \cdot \nabla\tau^2\, dx \\ &= B^{(i)}_{211} + B^{(i)}_{212}\,,\end{aligned}$$

where

$$\begin{aligned}B^{(i)}_{212} &\overset{h\to 0}{\rightarrow} -\int_\Omega \hat{\beta}_i (\rho_i + \delta) \log(\rho_i + \delta)\, u^{(i)} \cdot \nabla\tau^2\, dx \\ &\overset{\delta\to 0}{\rightarrow} -\int_\Omega \hat{\beta}_i \rho_i \log \rho_i u^{(i)} \cdot \nabla\tau^2\, dx\end{aligned}$$

and due to

$$\int \varphi u^{(i)} \cdot \nabla\tau^2\, dx \to \int_{\partial\Omega} \varphi u^{(i)} \cdot \vec{n}\, dS \text{ as } \tau \to 1$$

for a test function φ, the term $B^{(i)}_{212}$ finally vanishes in the limit since $u^{(i)}\big|_{\partial\Omega} = 0$.

$B^{(i)}_{211}$ is split like $B^{(i)}_{111m}$ into two parts from which the one with δ converges to zero as δ tends to zero and the other one is treated like $B^{(i)}_{1111m}$ with the aid of the DiPerna–Lions Lemma using that $\operatorname{div}\left(\rho_i u^{(i)}\right) = 0, i = 1, 2$, weakly, according to (3.10), to show that the term vanishes as h tends to zero.

To summarize, the terms with $\operatorname{div} u^{(i)}_m$ in (3.15) converge to zero as m tends to ∞ and τ tends to 1.

The terms with f_m and I_m, which are lower-order terms, vanish as well as $m \to \infty$, and we obtain finally

$$\int_\Omega (A_0 P(\rho^m))_i \, \Delta \left(\tau \Delta^{-1} \left((\rho_i^m - \rho_i)\, \tau\right)\right) \, dx$$
$$= \int_\Omega (A_0 P(\rho^m))_i \, (\rho_i^m - \rho_i)\, \tau^2 \, dx + o(1) \to 0 \text{ as } m \to \infty \text{ and } \tau \to 1\,.$$

With the monotonicity of the pressure (3.5) we can deduce from this idendity that

$$\rho_i^m \to \rho_i \text{ almost everywhere.}$$

Since we have imposed that $\|\rho_i^m\|_{L^p} \leq K$ uniformly, we can also conclude that

$$\rho_i^m \to \rho_i \text{ strongly in } L^r\,, 1 \leq r < p\,,$$

and we can pass to the limit also in the nonlinear pressure term $P_i(\rho^m)$, and it holds

$$\overline{P_i(\rho)} = P_i(\rho)\,,$$

such that (ρ, u) is a solution of the equations (3.1)–(3.3), (3.8). □

As mentioned above, the condition $\rho_i^m \in L^p(\Omega), p > 3$, corresponds to the case that $\gamma > 2$.

In the one-component case it is known (cf. [NN02], [NS04, p. 192]) that not only $\rho \in L^\gamma(\Omega)$, but that even better integrability properties for the density can be obtained, namely

$$\rho \in L^{s(\gamma)}(\Omega)\,, \text{ where } s(\gamma) = \begin{cases} 3\gamma - 3 & \text{if } \frac{3}{2} < \gamma \leq 3\,, \\ 2\gamma & \text{if } \gamma \geq 3\,. \end{cases}$$

Thus, if $\gamma > 2$, we can conclude that $\rho \in L^p(\Omega)$ for a $p > 3$.

Similarly, we can prove an a priori estimate for the densities in the case of mixtures.

In addition to (3.5) and (3.6), we have to impose the following condition on the pressure law $P_i(\rho)$: There exists a constant $C_1 > 0$ such that for $\rho = (\rho_1, \rho_2)^T$

$$\sum_{i=1}^{2} |P_i(\rho)| \geq C_1 |\rho|^\gamma - K\,. \tag{3.16}$$

Then we can derive the following

A priori estimate
Let Ω be a Lipschitz domain. Let (ρ, u) be a solution to the system (3.1)–(3.3), (3.8) as in Theorem 3.1 with the pressure fulfilling in addition condition (3.16). If for $i = 1, 2$ $u^{(i)} \in H_0^1(\Omega;\mathbb{R}^3)$ and $\rho_i \in L^\gamma(\Omega)$ with $\gamma > 2$, then $\rho_i \in L^p(\Omega)$ for a $p > 3$.

Proof:
To see this, we consider the momentum equation (3.2), which we write formally in the following way

$$D(P_i(\rho)) = D(\rho_i |u^{(i)}|^2) + D(z) + \text{ lower-order terms}\,,$$

where D symbolizes various first-order derivatives and z is an L^2-function.

By a duality argument (Nečas' lemma, cf. [MNRR96, Theorem 1.14] and [Neč66, Théorème 1]) we can conclude that for a suitably chosen $q \le 2$

$$\int_\Omega |\rho|^{q\gamma}\,dx \le \int_\Omega |\rho|^q |u|^{2q}\,dx + K\,,$$

where we used condition (3.16).

It turns out that the optimal choice for q is $q = \frac{3}{2} + \delta$ for a $\delta > 0$. (In fact, we have to apply Theorem 1.14 from [MNRR96] twice. Since we only know that $P_i(\rho) \in L^1(\Omega)$ at the beginning, we obtain the above inequality first for $q = \frac{3}{2} - \delta$. Then we can use the improved L^p-inclusion for $P_i(\rho)$ to choose a higher q.)
We obtain by applying Hölder's inequality and using the H_0^1-estimate for u

$$\begin{aligned}\int_\Omega |\rho|^{\left(\frac{3}{2}+\delta\right)\gamma}\,dx &\le \int_\Omega |\rho|^{\frac{3}{2}+\delta}|u|^{3+2\delta}\,dx + K\\ &\le K\left(\int_\Omega |\rho|^{\left(\frac{3}{2}+\delta\right)\frac{6}{3-2\delta}}\,dx\right)^{\frac{3-2\delta}{6}} + K\,.\end{aligned}$$

The exponent of ρ in the integral on the right-hand side equals

$$3 \cdot \frac{3+2\delta}{3-2\delta} = 3 + \delta'\,, \quad \delta' > 0\,.$$

The exponent of ρ in the integral on the left-hand side is greater than $3 + \delta$ since we have chosen $\gamma > 2$. Thus, the integral from the right-hand side can be absorbed on the left-hand side because it has the smaller exponent $\frac{3-2\delta}{6} < 1$. Finally, we have estimated

$$\int_\Omega |\rho|^{\left(\frac{3}{2}+\delta\right)\gamma}\,dx \le K\,,$$

which means in particular that $\rho_i \in L^p(\Omega)$ for a $p > 3$ because $\gamma > 2$. □

Remark: *In the evolutionary case it is not possible to extend the techniques for proving the compactness of solutions of the Navier–Stokes equations for compressible*

flow to the mixture model.
In the one-component case there appears a term coming from the time derivative of ρu *and from the convective term, which has a certain commutator structure. Thanks to this structure the convergence of this quantity can be obtained.*
In the case of mixtures, however, this technique does not work. In addition to terms of the form

$$\left(u_m^{(1)}\right)_j \left(\rho_1^m \left(u_m^{(1)}\right)_k \mathcal{R}_{jk}[\rho_1^m] - \rho_1^m \mathcal{R}_{jk}[\rho_1^m \left(u_m^{(1)}\right)_k]\right) ,$$

(analogously with 1 *replaced by* 2*), which can be treated like in the one-componet case using the div-curl lemma, there appear terms like e.g.*

$$\left(u_m^{(2)}\right)_j \left(\rho_2^m \left(u_m^{(2)}\right)_k \mathcal{R}_{jk}[\rho_1^m] - \rho_2^m \mathcal{R}_{jk}[\rho_1^m \left(u_m^{(1)}\right)_k]\right) .$$

Here, we use the notation from [Fei01] with $\mathcal{R}_{jk} =$ "$\partial_j \Delta^{-1} \partial_k$" *and summation convention (summing over* j, k *from* 1 *to* 3*.)*

The additional terms involving both ρ_1^m and ρ_2^m *do no longer have the commutator structure. Thus, the convergence of these quantities as* $m \to \infty$ *cannot be ensured. Therefore, up to now no compactness result is available in the unsteady case.*

Remark: *In our proof we need the assumption that the* ρ_i^m *are bounded in* $L^p(\Omega)$ *for a rather high* p*. In the one-component case this assumption was relaxed by Feireisl by introducing cut-off functions* $[\cdot]_L$ *(cf. [Fei01]). However, his technique can up to now not be applied to the case of mixtures because we cannot conclude from the relation*

$$\lim_{m\to\infty} \int_\Omega (A_0 P(\rho^m))_i \left([\rho_i^m]_L - [\rho_i]_L\right) \tau^2 \, dx = 0$$

the strong convergence of the densities.

Chapter 4

L^p-Estimates for the steady mixture model

In this chapter we present a new method of obtaining estimates for the densities ρ_i and the terms $\rho_i|u^{(i)}|^2$ for the steady mixture model including the convective terms. The results proved in this chapter are only regularity, but not existence results.

The method of proof was developed in [FGS04] for the one-component case for stationary flows. As mentioned in the introduction, the existence of weak solutions in the steady case of the Navier–Stokes equations for compressible isentropic flow

$$\begin{aligned} \operatorname{div}(\rho u) &= 0\,, \\ -\mu\Delta u - (\lambda+\mu)\nabla \operatorname{div} u + \rho u\cdot\nabla u &= -a\nabla\rho^\gamma + \rho f + g \end{aligned}$$

is only known for

$$\gamma > \frac{3}{2} \text{ if } \operatorname{curl} f = 0 \text{ and } \gamma > \frac{5}{3} \text{ if } \operatorname{curl} f \neq 0 \text{ (cf. [NN02])}\,.$$

Since air has the adiabatic constant $\gamma = \frac{7}{5}$, which is smaller than $\frac{3}{2}$, one is interested in obtaining the existence of solutions for smaller exponents γ.

In [FGS04] the authors present a new technique for estimating the density and the term $\rho|u|^2$, which works for $\gamma > \frac{5}{4}$. Thus, the physically important case of air is included. The estimates, in particular estimates for the term $\rho|u|^2$ in the space $L^{1+\delta}(\Omega)$, are important to apply the technique introduced by Feireisl ([Fei01]), which yields the compactness of the densities.

In this chapter of the thesis at hand, we adopt the techniques to the case of mixtures and present similar estimates for the system of equations:

For $i = 1, 2$

$$\operatorname{div}\left(\rho_i u^{(i)}\right) = 0\,, \tag{4.1}$$

$$\sum_{k=1}^{2} L_{ik} u^{(k)} + \operatorname{div}\left(\rho_i u^{(i)} \otimes u^{(i)}\right) = -\nabla P_i(\rho) + \rho_i f^{(i)} + I^{(i)}\,, \tag{4.2}$$

$$\rho_i \geq 0\,, \qquad \int_\Omega \rho_i \, dx = 1 \tag{4.3}$$

in a bounded domain $\Omega \subset \mathbb{R}^3$.
The operators L_{ik} are given by

$$L_{ik} = -\mu_{ik}\Delta - (\lambda_{ik} + \mu_{ik})\nabla \operatorname{div}$$

and fulfill the ellipticity condition

$$\sum_{i,k=1}^{2} \int_\Omega L_{ik} u^{(k)} \cdot u^{(i)} \, dx \geq c_0 \int_\Omega |\nabla u|^2 \, dx$$

for a constant $c_0 > 0$.

The interaction terms are as usual of the form

$$I^{(i)} = (-1)^{i+1} a \left(u^{(2)} - u^{(1)}\right)\,, a > 0\,.$$

We are interested in the case where the pressure $P(\rho)$ behaves like $|\rho|^\gamma$.

We assume that for a $\gamma > 1$ $P(\rho) = (P_1(\rho), P_2(\rho))^T$ fulfills the following *growth condition:* there is $K_1 > 0$ such that for all $\rho = (\rho_1, \rho_2)^T$ and $\hat{\rho} = (\hat{\rho}_1, \hat{\rho}_2)^T$

$$|P(\rho) - P(\hat{\rho})| \leq K_1 \left(|\rho|^{\gamma-1} + |\hat{\rho}|^{\gamma-1}\right) |\rho - \hat{\rho}|\,. \tag{4.4}$$

Moreover, we have to impose that for $i = 1, 2$

$$P_i(\rho) \geq \tilde{C}_i \rho_i^\gamma\,, \quad \tilde{C}_i > 0\,. \tag{4.5}$$

Remark: *We could also assume condition (4.5) with different exponents γ_i for $i = 1, 2$, which should fulfill the same bounds which we impose for γ below.*

Here, we deal with the case that $\frac{5}{4} < \gamma \leq 5$ as *estimates* are concerned. We have to emphasize that we prove some regularity for solutions of the equations (4.1)–(4.3) under the *assumption* that we have solutions which fulfill certain estimates. However, in the general case of the system (4.1)–(4.3) it is not clear how to obtain such solutions.

No boundary conditions are imposed since we present only local estimates.

For simplicity, we assume that

$$f^{(i)} \in L^{\infty}(\Omega; \mathbb{R}^3) .$$

We prove the following two theorems.

Theorem 4.1 *Assume that* $f^{(i)} \in L^{\infty}$. *Let* $\frac{5}{4} < \gamma \leq 5$. *Let* $P_i(\rho)$ *fulfill the conditions (4.4) and (4.5). Let* $(\rho, u), \rho = (\rho_1, \rho_2)^T, u = (u^{(1),T}, u^{(2),T})^T$, *be a solution of the momentum equation (4.2) with* $u^{(i)} \in H^1(\Omega; \mathbb{R}^3), \rho_i \in L^{\gamma}_{loc}(\Omega), \rho_i \geq 0, \rho_i |u^{(i)}|^2 \in L^{1+\delta}_{loc}(\Omega)$. *Then*

$$\rho_i \in L^q_{loc}(\Omega) \text{ with } q = \frac{6\gamma^2}{5 + 2\gamma} .$$

The L^q_{loc}*-estimate for* ρ_i *is uniform on compact subdomains with respect to* $\|u\|_{H^1}$, $\|\rho_i\|_{L^{\gamma}_{loc}}$ *and* $\|\rho_i |u^{(i)}|^2\|_{L^{1+\delta}_{loc}}$.

Theorem 4.2 *The same assertion as in Theorem 4.1 holds with the additional statement that the* L^q_{loc}*-estimate of* ρ_i *is uniform with respect to* $\|u\|_{H^1}, \|\rho_i\|_{L^{\gamma}_{loc}}$ *and* $\|\rho_i |u^{(i)}|^2\|_{L^1_{loc}}$.

In the one-component case, it is known that (cf. [NN02], [NS04])

$$\rho \in L^{s(\gamma)}(\Omega) , \text{ where } s(\gamma) = \begin{cases} 3(\gamma - 1) , & \frac{3}{2} < \gamma \leq 3 , \\ 2\gamma , & \gamma \geq 3 . \end{cases}$$

Since

$$\frac{6\gamma^2}{5 + 2\gamma} \geq 3(\gamma - 1) \text{ for } \gamma \leq \frac{5}{3} ,$$

the estimates presented in [FGS04] are an improvement of known estimates for $\frac{5}{4} < \gamma \leq \frac{5}{3}$.

We think that it will turn out useful to apply our technique of estimating also in the case of mixtures.

The proof of the theorems is organized as follows:

In Proposition 4.1 and 4.2 we prove local weighted estimates for $P_i(\rho)$ and $\rho_i |u^{(i)}|^2$. In fact, we obtain estimates for $P_i(\rho)$ in $L^1_{loc}(\Omega)$ with the weight $|x - x_0|^{-1}$ and, moreover, for $P_i(\rho)$ and $\rho_i |u^{(i)}|^2$ in $L^1_{loc}(\Omega)$ with the weight $|x - x_0|^{-1+\varepsilon}$ for a small $\varepsilon > 0$. We do not need the assumption $\rho_i |u^{(i)}|^2 \in L^{1+\delta}_{loc}(\Omega)$ for these estimates, an L^1_{loc}-inclusion is sufficient.

These estimates are then used to estimate locally $\rho_i^\gamma|u^{(i)}|$. Under the assumption $\rho_i|u^{(i)}|^2 \in L^{1+\delta}_{loc}(\Omega)$ (as in Theorem 4.1), we prove in Proposition 4.3 that $P_i(\rho)|u^{(i)}| \in L^1_{loc}(\Omega)$ and thus $\rho_i^\gamma|u^{(i)}| \in L^1_{loc}(\Omega)$.

In Proposition 4.4 we derive the estimate $\rho_i^\gamma|u^{(i)}| \in L^1_{loc}(\Omega)$ under the assumption $\rho_i|u^{(i)}|^2 \in L^1_{loc}(\Omega)$ only (as in Theorem 4.2).

The key idea in this approach is to manipulate the convective term in a way that we can use a higher L^p-integrability of the term $\nabla\Delta^{-1}(\rho_i|u^{(i)}|^2)$, which follows from the weighted estimate for $\rho_i|u^{(i)}|^2$ (Proposition 4.2).

With the help of the local estimate for $\rho_i^\gamma|u^{(i)}|$ we can then conclude by Hölder's inequality a better integrability for $\rho_i|u^{(i)}|^2$, namely $\rho_i|u^{(i)}|^2 \in L_{loc}^{\frac{6\gamma}{5+2\gamma}}(\Omega)$.
Via the momentum equation the estimate claimed for ρ_i in Theorem 4.1 and 4.2 then follows.

4.1 Local weighted estimates for $P_i(\rho)$ and $\rho_i|u^{(i)}|^2$

In this section we prove weighted estimates for the pressure $P_i(\rho)$ and the terms $\rho_i|u^{(i)}|^2$, which are the first steps needed for our approach.

Proposition 4.1 *Let (ρ, u), $\rho = (\rho_1, \rho_2)^T, u = (u^{(1),T}, u^{(2),T})^T$, be a solution of the equation (4.2) with $u^{(i)} \in H^1(\Omega;\mathbb{R}^3), P_i(\rho) \in L^1_{loc}(\Omega)$ and $\rho_i|u^{(i)}|^2 \in L^1_{loc}(\Omega)$. Then the following estimate holds in $\Omega_0 \subset\subset \Omega$:*

$$\int_{\Omega_0} \frac{P_i(\rho)}{|x-x_0|}\,dx \le K \tag{4.6}$$

for arbitrary $x_0 \in \Omega$. The estimate is uniform with respect to $\|u\|_{H^1}, \|\rho_i|u^{(i)}|^2\|_{L^1_{loc}}$, $\|P_i(\rho)\|_{L^1_{loc}}$.

Proof:
Testing in (4.2) by $\frac{x-x_0}{|x-x_0|}\tau^2$ where $\tau \in \mathcal{D}(\Omega)$ is a localization function in Ω leads to

for $i = 1, 2$

$$\sum_{k=1}^{2}\Big\{\mu_{ik}\int_\Omega \nabla u^{(k)} : \nabla\left(\frac{x-x_0}{|x-x_0|}\right)\tau^2\,dx + \mu_{ik}\int_\Omega \nabla u^{(k)} : \frac{x-x_0}{|x-x_0|}\nabla\tau^2\,dx$$
$$+(\lambda_{ik}+\mu_{ik})\int_\Omega \operatorname{div} u^{(k)}\frac{2}{|x-x_0|}\tau^2\,dx + (\lambda_{ik}+\mu_{ik})\int_\Omega \operatorname{div} u^{(k)}\frac{x-x_0}{|x-x_0|}\cdot\nabla\tau^2\,dx\Big\}$$
$$-\int_\Omega\sum_{j,l=1}^{3}\rho_i u_l^{(i)}u_j^{(i)}\partial_l\left(\frac{(x-x_0)_j}{|x-x_0|}\right)\tau^2\,dx - \int_\Omega\sum_{j,l=1}^{3}\rho_i u_l^{(i)}u_j^{(i)}\frac{(x-x_0)_j}{|x-x_0|}\partial_l\tau^2\,dx$$
$$=\int_\Omega P_i(\rho)\frac{2}{|x-x_0|}\tau^2\,dx + \int_\Omega P_i(\rho)\frac{x-x_0}{|x-x_0|}\cdot\nabla\tau^2\,dx$$
$$+\int_\Omega \rho_i f^{(i)}\cdot\frac{x-x_0}{|x-x_0|}\tau^2\,dx + \int_\Omega I^{(i)}\cdot\frac{x-x_0}{|x-x_0|}\tau^2\,dx\,. \tag{4.7}$$

The term $\frac{x-x_0}{|x-x_0|}\nabla\tau^2$ has to be understood as a 3×3 matrix (dyadic product).

The sum $\sum_{k=1}^{2}$ over the four terms is bounded because $u^{(i)} \in H^1(\Omega;\mathbb{R}^3), i = 1, 2, \nabla\left(\frac{x-x_0}{|x-x_0|}\right) \in L^2(\Omega;\mathbb{R}^{3\times 3})$ in three dimensions.

The last term on the left-hand side is bounded as well because of the assumption $\rho_i|u^{(i)}|^2 \in L^1_{loc}(\Omega)$.

The same holds for the terms including $f^{(i)}$ and $I^{(i)}$ because $f^{(i)} \in L^\infty(\Omega;\mathbb{R}^3)$ and the interaction terms are of the form $I^{(i)} = (-1)^{i+1}a(u^{(2)} - u^{(1)})$ and we have due to the assumptions $u^{(i)} \in L^6(\Omega;\mathbb{R}^3), i = 1, 2$.

We consider the last but one term on the left-hand side in (4.7):

$$-\int_\Omega\sum_{j,l=1}^{3}\rho_i u_l^{(i)}u_j^{(i)}\partial_l\left(\frac{(x-x_0)_j}{|x-x_0|}\right)\tau^2\,dx$$
$$= -\int_\Omega \frac{\rho_i|u^{(i)}|^2}{|x-x_0|}\tau^2\,dx + \int_\Omega \frac{\rho_i\left(u^{(i)}\cdot(x-x_0)\right)^2}{|x-x_0|^3}\tau^2\,dx$$
$$\le 0\,.$$

Therewith, we obtain finally from (4.7)

$$2\int_\Omega \frac{P_i(\rho)}{|x-x_0|}\tau^2\,dx \;\le\; K - \int_\Omega P_i(\rho)\frac{x-x_0}{|x-x_0|}\cdot\nabla\tau 2\tau\,dx$$
$$\le\; K + \delta\int_\Omega \frac{P_i(\rho)}{|x-x_0|}\tau^2\,dx + K\int_\Omega P_i(\rho)|x-x_0||\nabla\tau|^2\,dx\,.$$

The first integral can be absorbed on the left-hand side, the second one is bounded because of the assumption that $P_i(\rho) \in L^1_{loc}(\Omega)$. Thus, we obtain

$$\int_\Omega \frac{P_i(\rho)}{|x-x_0|}\tau^2\,dx \le K\,,$$

which was claimed in the proposition. □

Proposition 4.2 *Let $(\rho, u), \rho = (\rho_1, \rho_2)^T, u = (u^{(1),T}, u^{(2),T})^T$, be a solution to the equation (4.2) with $u^{(i)} \in H^1(\Omega; \mathbb{R}^3), P_i(\rho) \in L^1_{loc}(\Omega)$ and $\rho_i |u^{(i)}|^2 \in L^1_{loc}(\Omega)$. Then the following estimates hold in $\Omega_0 \subset\subset \Omega$ for $\varepsilon > 0$ small and $x_0 \in \Omega_0$ arbitrary:*

$$\int_{\Omega_0} \frac{P_i(\rho)}{|x - x_0|^{1-\varepsilon}}\, dx \ \leq \ K\,, \tag{4.8}$$

$$\varepsilon \int_{\Omega_0} \frac{\rho_i |u^{(i)}|^2}{|x - x_0|^{1-\varepsilon}}\, dx \ \leq \ K\,. \tag{4.9}$$

The estimate is uniform with respect to $\|u\|_{H^1}, \|\rho_i |u^{(i)}|^2\|_{L^1_{loc}}, \|P_i(\rho)\|_{L^1_{loc}}$.

Proof:
This time we test in (4.2) by $\frac{x-x_0}{|x-x_0|^{1-\varepsilon}}\, \tau^2\, dx$ and obtain for $i = 1, 2$

$$\begin{aligned}
&(2+\varepsilon) \int_\Omega \frac{P_i(\rho)}{|x-x_0|^{1-\varepsilon}}\, \tau^2\, dx + \int_\Omega \sum_{j,l=1}^{3} \rho_i u_l^{(i)} u_j^{(i)} \partial_l \left(\frac{(x-x_0)_j}{|x-x_0|} \right) \tau^2\, dx \\
&= \sum_{k=1}^{2} \Big\{ \mu_{ik} \int_\Omega \nabla u^{(k)} : \nabla \left(\frac{x-x_0}{|x-x_0|^{1-\varepsilon}} \right) \tau^2\, dx + \mu_{ik} \int_\Omega \nabla u^{(k)} : \frac{x-x_0}{|x-x_0|^{1-\varepsilon}}\, \nabla \tau^2\, dx \\
&\qquad + (\lambda_{ik} + \mu_{ik}) \int_\Omega \operatorname{div} u^{(k)} \frac{2+\varepsilon}{|x-x_0|^{1-\varepsilon}}\, \tau^2\, dx \\
&\qquad + (\lambda_{ik} + \mu_{ik}) \int_\Omega \operatorname{div} u^{(k)} \frac{x-x_0}{|x-x_0|^{1-\varepsilon}} \cdot \nabla \tau^2\, dx \Big\} \\
&- \int_\Omega P_i(\rho) \frac{x-x_0}{|x-x_0|^{1-\varepsilon}} \cdot \nabla \tau^2\, dx - \int_\Omega \rho_i f^{(i)} \cdot \frac{x-x_0}{|x-x_0|^{1-\varepsilon}}\, \tau^2\, dx \\
&- \int_\Omega I^{(i)} \cdot \frac{x-x_0}{|x-x_0|^{1-\varepsilon}}\, \tau^2\, dx - \int_\Omega \sum_{j,l=1}^{3} \rho_i u_l^{(i)} u_j^{(i)} \frac{(x-x_0)_j}{|x-x_0|^{1-\varepsilon}}\, \partial_l \tau^2\, dx\,.
\end{aligned} \tag{4.10}$$

Here, $\frac{x-x_0}{|x-x_0|^{1-\varepsilon}} \nabla \tau^2$ is as above a 3×3 matrix.

The sum $\sum_{k=1}^{2}$ over the four terms is bounded due to $u^{(i)} \in H^1(\Omega; \mathbb{R}^3), i = 1, 2$, and the integrability properties of the functions τ and $\frac{x-x_0}{|x-x_0|^{1-\varepsilon}}$ in three dimensions.

The next term can be estimated as follows

$$\begin{aligned}
&\left| \int_\Omega P_i(\rho) \frac{x-x_0}{|x-x_0|^{1-\varepsilon}} \cdot \nabla \tau 2\tau\, dx \right| \\
&\leq \delta \int_\Omega \frac{P_i(\rho)}{|x-x_0|^{1-\varepsilon}}\, \tau^2\, dx + K \int_\Omega P_i(\rho) |x-x_0|^{1+\varepsilon} |\nabla \tau|^2\, dx\,,
\end{aligned}$$

where the first integral can be absorbed on the left-hand side and the last one is bounded due to $P_i(\rho) \in L^1_{loc}(\Omega)$.

The terms including $f^{(i)}$ and $I^{(i)}$ are bounded as well.

We consider now the second term on the left-hand side of (4.10):

$$\begin{aligned}
&\int_\Omega \sum_{j,l=1}^{3} \rho_i u_l^{(i)} u_j^{(i)} \partial_l \left(\frac{(x-x_0)_j}{|x-x_0|^{1-\varepsilon}} \right) \tau^2 \, dx \\
&= \int_\Omega \sum_{j,l=1}^{3} \rho_i u_l^{(i)} u_j^{(i)} \left(\delta_{lj} \frac{1}{|x-x_0|^{1-\varepsilon}} + (\varepsilon - 1)(x-x_0)_j \frac{(x-x_0)_l}{|x-x_0|^{3-\varepsilon}} \right) \tau^2 \, dx \\
&= \int_\Omega \frac{\rho_i |u^{(i)}|^2}{|x-x_0|^{1-\varepsilon}} \tau^2 \, dx + (\varepsilon - 1) \int_\Omega \frac{\rho_i \left(u^{(i)} \cdot (x-x_0) \right)^2}{|x-x_0|^{3-\varepsilon}} \tau^2 \, dx \\
&\geq \varepsilon \int_\Omega \frac{\rho_i |u^{(i)}|^2}{|x-x_0|^{1-\varepsilon}} \tau^2 \, dx \, .
\end{aligned}$$

Here, δ_{lj} is the Kronecker symbol; it is one for $l = j$ and zero otherwise.

After these considerations we can estimate the last term on the right-hand side of (4.10) as follows:

$$\begin{aligned}
&\left| \int_\Omega \sum_{j,l=1}^{3} \rho_i u_l^{(i)} u_j^{(i)} \frac{(x-x_0)_j}{|x-x_0|^{1-\varepsilon}} \, \partial_l \tau^2 \, dx \right| \\
&\leq \tilde{\delta} \int_\Omega \frac{\rho_i |u^{(i)}|^2}{|x-x_0|^{1-\varepsilon}} \tau^2 \, dx + K \int_\Omega \rho_i |u^{(i)}|^2 |x-x_0|^{1+\varepsilon} |\nabla \tau|^2 \, dx \, .
\end{aligned}$$

Here, we can absorb the first term on the left-hand side of (4.10), choosing for instance $\tilde{\delta} = \frac{\varepsilon}{2}$. The last integral is bounded due to the assumption $\rho_i|u^{(i)}|^2 \in L^1_{loc}(\Omega)$.

Thus, we obtain finally from (4.10)

$$(2 + \varepsilon - \delta) \int_\Omega \frac{P_i(\rho)}{|x-x_0|^{1-\varepsilon}} \tau^2 \, dx + (\varepsilon - \tilde{\delta}) \int_\Omega \frac{\rho_i |u^{(i)}|^2}{|x-x_0|^{1-\varepsilon}} \tau^2 \, dx \leq K \, ,$$

which is nothing else than what was claimed in the proposition. □

From this estimate we can derive the following

Corollary 4.1 *The solution of the problem*

$$\begin{aligned}
-\Delta w^{(i)} &= \rho_i |u^{(i)}|^2 \qquad in \ \Omega \, , \\
w^{(i)} &= 0 \qquad on \ \partial\Omega
\end{aligned}$$

belongs to $L^p_{loc}(\Omega)$ *for all* p *with* $1 \le p < \infty$.

Moreover, if $\rho_i |u^{(i)}|^2 \in L^{1+\varepsilon}_{loc}(\Omega)$*, then* $\nabla w^{(i)} \in L^{2+\varepsilon}_{loc}(\Omega; \mathbb{R}^3)$.

Proof:
The first statement is a consequence of the weighted estimate $\int_{\Omega_0} \frac{\rho_i |u^{(i)}|^2}{|x-x_0|^{1-\varepsilon}}\, dx \le K$.

We present an elemantary proof of this consequence. More general, we prove for a domain $\Omega \subset \mathbb{R}^n, n \ge 3$, the following assertion:

Let $\int_\Omega \frac{\rho_i |u^{(i)}|^2}{|x-x_0|^{n-2-\varepsilon}}\, dx \le K$. Then the solution $w^{(i)}$ of the problem

$$-\Delta w^{(i)} = \rho_i |u^{(i)}|^2 \quad \text{in } \Omega\,, \tag{4.11}$$

$$w^{(i)} = 0 \quad \text{on } \partial\Omega \tag{4.12}$$

belongs to the space $L^p(\Omega)$ for all p with $1 \le p < \infty$.

We test the equation (4.11) by $\frac{w^{(i)}}{\sqrt[s]{1+|w^{(i)}|^s}} \frac{1}{|x-x_0|^{n-2-\varepsilon}}$. (More precisely, we have to test by
$\frac{w^{(i)}}{\sqrt[s]{1+|w^{(i)}|^s}} \frac{1}{(|x-x_0|+h)^{n-2-\varepsilon}}$ with some positive h and then perform the limit process as $h \to 0$, but this is only a technical procedure.)

Due to the weighted estimate for $\rho_i |u^{(i)}|^2$, the right-hand side in what we get, namely,

$$\int_\Omega \rho_i |u^{(i)}|^2 \frac{w^{(i)}}{\sqrt[s]{1+|w^{(i)}|^s}} \frac{1}{|x-x_0|^{n-2-\varepsilon}}\, dx\,,$$

is bounded.

Using $\left(\frac{\xi}{\sqrt[s]{1+|\xi|^s}}\right)' = \frac{1}{(1+|\xi|^s)^{\frac{1+s}{s}}}$, we estimate

$$\int_\Omega \frac{|\nabla w^{(i)}|^2}{(1+|w^{(i)}|^s)^{\frac{1+s}{s}}} \frac{1}{|x-x_0|^{n-2-\varepsilon}}\, dx + \int_\Omega \nabla F(w^{(i)}) \nabla \left(\frac{1}{|x-x_0|^{n-2-\varepsilon}}\right) dx \le K\,,$$

where F is a primitive function of $\frac{\xi}{\sqrt[s]{1+|\xi|^s}}$, which obviously has linear growth.

The second integral in this inequality, which is after integration by parts
$-\int_\Omega F(w^{(i)}) \Delta \frac{1}{|x-x_0|^{n-2-\varepsilon}}\, dx$, is nonnegative so that we can neglect it. Using the fact that $(1+\xi^s)^{\frac{1+s}{s}} \sim c(1+\xi)^{s+1}$ we obtain

$$c \int_\Omega \frac{|\nabla w^{(i)}|^2}{(1+|w^{(i)}|)^{1+s}} \frac{1}{|x-x_0|^{n-2-\varepsilon}}\, dx \le K\,.$$

Rewriting the first term in this integral gives

$$c\int_\Omega \left|\nabla |w^{(i)}|^{\frac{1-s}{2}}\right|^2 \frac{1}{|x-x_0|^{n-2-\varepsilon}}\,dx \le K\,.$$

This estimate, which is the limit case in the theory of Morrey spaces, gives

$$|w^{(i)}|^{\frac{1-s}{2}} \in L^p(\Omega) \quad \forall p \text{ with } 1 \le p < \infty\,.$$

This proves the first statement of the corollary.

The second statement can be proven by using interpolation and the identity

$$\int_\Omega |\nabla w^{(i)}|^2\,dx = -\int_\Omega w^{(i)}\Delta w^{(i)}\,dx\,.$$

□

4.2 A local estimate for $\rho_i^\gamma|u^{(i)}|$ under the assumption that $\rho_i|u^{(i)}|^2 \in L^{1+\delta}_{loc}(\Omega)$

In addition to the usual assumptions we impose in this chapter that $\rho_i|u^{(i)}|^2 \in L^{1+\delta}_{loc}(\Omega)$. If we have this information, the proof of the estimate $\rho_i^\gamma|u^{(i)}| \in L^1_{loc}(\Omega)$ is easier. In the next section, we present the proof under the assumption that $\rho_i|u^{(i)}|^2 \in L^1_{loc}(\Omega)$ only.

Proposition 4.3 *Let $(\rho,u), \rho=(\rho_1,\rho_2)^T, u=(u^{(1),T},u^{(2),T})^T$, be a solution to the equation (4.2) with $u^{(i)} \in H^1(\Omega;\mathbb{R}^3), P_i(\rho) \in L^1_{loc}(\Omega)$ and $\rho_i|u^{(i)}|^2 \in L^{1+\delta}_{loc}(\Omega)$. Then the following estimate holds in $\Omega_0 \subset\subset \Omega$:*

$$\int_{\Omega_0} P_i(\rho)|u^{(i)}|\,dx \le K\,.$$

Here, K depends on $f^{(i)}$ and $\|u\|_{H^1}, \|\rho_i|u^{(i)}|^2\|_{L^{1+\delta}_{loc}}, \|\rho_i\|_{L^\gamma_{loc}}$.

Proof:
Consider the following auxiliary problem

$$\begin{aligned} -\Delta z^{(i)} &= |u^{(i)}| && \text{in } \Omega\,,\\ z^{(i)} &= 0 && \text{on } \partial\Omega\,. \end{aligned}$$

Since $u^{(i)} \in L^6(\Omega;\mathbb{R}^3)$, the second derivatives of $z^{(i)}$ are in L^6. Thus, the first derivatives are in L^∞. The third derivatives belong to L^2.

We use in (4.2) the test function $\tau\nabla z^{(i)}$, where τ is a smooth localization function, and obtain for $i = 1, 2$

$$\sum_{k=1}^{2}\Big\{\mu_{ik}\int_\Omega \nabla u^{(k)} : \nabla^2 z^{(i)}\tau\,dx + \mu_{ik}\int_\Omega \nabla u^{(k)} : \nabla z^{(i)}\nabla\tau\,dx$$
$$+(\lambda_{ik}+\mu_{ik})\int_\Omega \operatorname{div} u^{(k)}\Delta z^{(i)}\tau\,dx + (\lambda_{ik}+\mu_{ik})\int_\Omega \operatorname{div} u^{(k)}\nabla z^{(i)}\cdot\nabla\tau\,dx\Big\}$$
$$-\int_\Omega\sum_{j,l=1}^{3}\rho_i u_l^{(i)}u_j^{(i)}\partial_l\left(\tau\partial_j z^{(i)}\right)\,dx = -\int_\Omega P_i(\rho)|u^{(i)}|\tau\,dx$$
$$+\int_\Omega P_i(\rho)\nabla z^{(i)}\cdot\nabla\tau\,dx + \int_\Omega \rho_i f^{(i)}\cdot\nabla z^{(i)}\tau\,dx + \int_\Omega I^{(i)}\cdot\nabla z^{(i)}\tau\,dx\,,$$

whereby the expression $\nabla z^{(i)}\nabla\tau$ in the second term has to be understood as a 3×3 matrix.

Using the L^p-inclusions of the function $z^{(i)}$ and its derivatives as well as those of $u^{(i)}$, $f^{(i)}$ and $I^{(i)}$ we can estimate

$$\begin{aligned}
\int_\Omega P_i(\rho)|u^{(i)}|\tau\,dx &\le K + \int_\Omega\sum_{j,l=1}^{3}\rho_i u_l^{(i)}u_j^{(i)}\tau\partial_l\partial_j z^{(i)}\,dx + \int_\Omega\sum_{j,l=1}^{3}\rho_i u_l^{(i)}u_j^{(i)}\partial_l\tau\partial_j z^{(i)}\,dx\\
&\le K + \int_\Omega\sum_{j,l=1}^{3}\rho_i u_l^{(i)}u_j^{(i)}\tau\partial_l\partial_j z^{(i)}\,dx \quad \text{due to } \rho_i|u^{(i)}|^2\in L^{1+\delta}_{loc}(\Omega)\,,\\
&\quad (L^1_{loc}(\Omega) \text{ would be sufficient because the first derivatives of } z^{(i)}\\
&\quad\ \text{are bounded, as well as } \nabla\tau\,.)\\
&\le K + \left|\int_\Omega\sum_{j,l=1}^{3}\nabla\Delta^{-1}\left(\rho_i u_l^{(i)}u_j^{(i)}\right)\cdot\nabla\left(\partial_l\partial_j z^{(i)}\right)\tau\,dx\right|\\
&\quad + \left|\int_\Omega\sum_{j,l=1}^{3}\nabla\Delta^{-1}\left(\rho_i u_l^{(i)}u_j^{(i)}\right)\cdot\nabla\tau\partial_l\partial_j z^{(i)}\,dx\right|\,.
\end{aligned}$$

Here, we understand the operator Δ^{-1} as solving the Laplace equation in Ω with zero boundary values.

We consider now the (more difficult) first term (the second term is less complicated):

$$\left|\int_\Omega\sum_{j,l=1}^{3}\nabla\Delta^{-1}\left(\rho_i u_l^{(i)}u_j^{(i)}\right)\cdot\nabla\left(\partial_l\partial_j z^{(i)}\right)\tau\,dx\right| \le K\left\|\sum_{j,l=1}^{3}\nabla\Delta^{-1}\left(\rho_i u_l^{(i)}u_j^{(i)}\right)\right\|_{L^2}$$

because $\nabla^3 z^{(i)}\in L^2, \tau\in L^\infty$.

The Morrey estimates which we have obtained in the preceding section serve us here by ensuring better properties of the terms $\nabla\Delta^{-1}\left(\rho_i|u^{(i)}|^2\right)$ (even though not for the quantities $\rho_i|u^{(i)}|^2$ themselves).

We use now integration by parts

$$\begin{aligned}\left(\int_\Omega |\nabla w|^2\,dx\right)^{\frac{1}{2}} &= \left(\int_\Omega \nabla w\cdot\nabla w\right)^{\frac{1}{2}}\\ &= \left(-\int_\Omega w\Delta w\,dx\right)^{\frac{1}{2}}\\ &\le \|\Delta w\|^{\frac{1}{2}}_{L^p}\|w\|^{\frac{1}{2}}_{L^{\frac{p-1}{p}}}\end{aligned}$$

to estimate the L^2-norm further for $j,l = 1,2,3$

$$\left\|\nabla\Delta^{-1}\left(\rho_i u_l^{(i)}u_j^{(i)}\right)\right\|_{L^2} \le K\|\rho_i u_l^{(i)}u_j^{(i)}\|^{\frac{1}{2}}_{L^{1+\delta}}\|\Delta^{-1}(\rho_i u_l^{(i)}u_j^{(i)})\|^{\frac{1}{2}}_{L^{\frac{1+\delta}{\delta}}}\,.$$

Since $\rho_i|u^{(i)}|^2 \in L^{1+\delta}_{loc}(\Omega)$, the first norm is bounded.

According to Corollary 4.1 it holds that $\Delta^{-1}(\rho_i|u^{(i)}|^2) \in L^p_{loc}(\Omega)$ for all $p, 1 \le p < \infty$. Due to the monotonicity of Δ^{-1} we have that $|\Delta^{-1}(\rho_i u_l^{(i)}u_j^{(i)})| \le \Delta^{-1}(\rho_i|u^{(i)}|^2)$. Therefore, the norm $\|\Delta^{-1}(\rho_i u_l^{(i)}u_j^{(i)})\|_{L^{\frac{1+\delta}{\delta}}}$ is bounded as well for $j,l = 1,2,3$.

Thus, we have proved under the additional assumption $\rho_i|u^{(i)}|^2 \in L^{1+\delta}_{loc}(\Omega)$ that

$$\int_\Omega P_i(\rho)|u^{(i)}|\tau\,dx \le K\,.$$

□

With the help of condition (4.5) for the pressure, from this estimate follows that $\rho_i^\gamma|u^{(i)}| \in L^1_{loc}(\Omega)$.

4.3 A local estimate for $\rho_i^\gamma|u^{(i)}|$ under the assumption that $\rho_i|u^{(i)}|^2 \in L^1_{loc}(\Omega)$ only

In the preceding section, we imposed an estimate for the term $\rho_i|u^{(i)}|^2$ in $L^{1+\delta}_{loc}(\Omega)$. In the classical case for one-component compressible fluids, such an estimate is not available for small γ. In the case $\gamma = \frac{3}{2}$, however, one has an estimate for $\rho|u|^2$ in L^1. Therefore, we would like to show that we can achieve an estimate for $\rho_i^\gamma|u^{(i)}|$ even under only the premise that $\rho_i|u^{(i)}|^2 \in L^1_{loc}(\Omega)$.

We prove the following

Proposition 4.4 *Let (ρ, u) be a solution of the equation (4.2) with $u^{(i)} \in H^1(\Omega;\mathbb{R}^3)$, $\rho_i \in L^\gamma_{loc}(\Omega)$ and $\rho_i|u^{(i)}|^2 \in L^1_{loc}(\Omega)$. Then the following estimate holds in $\Omega_0 \subset\subset \Omega$:*

$$\int_{\Omega_0} \rho_i^\gamma |u^{(i)}|\, dx \leq K$$

with K depending on $f^{(i)}, \|u\|_{H^1}, \|\rho_i\|_{L^\gamma_{loc}}, \|\rho_i|u^{(i)}|^2\|_{L^1_{loc}}$.

Proof:
We will qualitatively use the $L^{1+\delta}$-norm of $\rho_i|u^{(i)}|^2$, but we will only need that the quantity is bounded in L^1 locally.

The first part of the proof is the same as in the proof of Proposition 4.3.

We obtain

$$\begin{aligned}\int_\Omega P_i(\rho)|u^{(i)}|\tau\, dx \quad &\leq \quad K + \left|\int_\Omega \sum_{j,l=1}^3 \nabla\Delta^{-1}\left(\rho_i u_l^{(i)} u_j^{(i)}\right) \cdot \nabla\left(\partial_l\partial_j z^{(i)}\right)\tau\, dx\right| \\ &\quad + \left|\int_\Omega \sum_{j,l=1}^3 \nabla\Delta^{-1}\left(\rho_i u_l^{(i)} u_j^{(i)}\right) \cdot \nabla\tau\partial_l\partial_j z^{(i)}\, dx\right| \qquad (4.13)\end{aligned}$$

and investigate (since more difficult) the first integral in (4.13):

$$\left|\int_\Omega \sum_{j,l=1}^3 \nabla\Delta^{-1}\left(\rho_i u_l^{(i)} u_j^{(i)}\right) \cdot \nabla\left(\partial_l\partial_j z^{(i)}\right)\tau\, dx\right| \quad \leq \quad K\left\|\sum_{j,l=1}^3 \nabla\Delta^{-1}\left(\rho_i u_l^{(i)} u_j^{(i)}\right)\right\|_{L^2}$$

due to $\nabla^3 z^{(i)} \in L^2$.

Further, for j, l from 1 to 3

$$\left\|\nabla\Delta^{-1}\left(\rho_i u_l^{(i)} u_j^{(i)}\right)\right\|_{L^2} \leq K\left\|\rho_i|u^{(i)}|^2\right\|^{\frac{1}{2}}_{L^{1+\delta}}\left\|\Delta^{-1}\left(\rho_i|u^{(i)}|^2\right)\right\|^{\frac{1}{2}}_{L^{\frac{1+\delta}{\delta}}},$$

where we have used the monotonicity of Δ^{-1} in the last term. This norm is bounded as $\Delta^{-1}(\rho_i|u^{(i)}|^2) \in L^p_{loc}$ for all $p, 1 \leq p < \infty$, according to Corollary 4.1.

The second integral in (4.13) is even easier to treat. Thus, we obtain from (4.13) by using condition (4.5) for the pressure law

$$\int_\Omega \rho_i^\gamma |u^{(i)}| \tau \, dx \leq K \left\|\rho_i |u^{(i)}|^2\right\|^{\frac{1}{2}}_{L^{1+\delta}} + K \,. \tag{4.14}$$

We analyze the $L^{1+\delta}$-norm further:

$$\begin{aligned}\left(\int_\Omega \left(\rho_i|u^{(i)}|^2\right)^{1+\delta} dx\right)^{\frac{1}{2(1+\delta)}} &= \left(\int_\Omega \left(\rho_i|u^{(i)}|^{\frac{1}{\gamma}}|u^{(i)}|^{2-\frac{1}{\gamma}}\right)^{1+\delta} dx\right)^{\frac{1}{2(1+\delta)}} \\ &\leq \left(\int_\Omega \rho_i^\gamma |u^{(i)}| \, dx\right)^{\frac{1}{2\gamma}} \left(\int_\Omega |u^{(i)}|^{\left(2-\frac{1}{\gamma}\right)(1+\delta)\frac{\gamma}{\gamma-1-\delta}} dx\right)^{\frac{\gamma-1-\delta}{2\gamma(1+\delta)}} .\end{aligned}$$

If the exponent of $|u^{(i)}|$ in the second integral fulfills

$$\left(2 - \frac{1}{\gamma}\right)(1+\delta)\frac{\gamma}{\gamma - 1 - \delta} \leq 6 \,, \text{ i.e. } \gamma \geq \frac{5+5\delta}{4-2\delta} \,,$$

which is the case for $\gamma > \frac{5}{4}$ as imposed, the integral is bounded due to $u^{(i)} \in H^1(\Omega;\mathbb{R}^3)$.

Here is the reason why we have to impose the restriction on γ.

The first integral can then be absorbed on the left-hand side of (4.14) due to the smaller exponent, and we obtain the estimate

$$\int_\Omega \rho_i^\gamma |u^{(i)}| \tau \, dx \leq K \,.$$

□

4.4 Proof of Theorem 4.1 and 4.2

The proof of Theorem 4.1 and 4.2 follows from Proposition 4.3 or Proposition 4.4, respectively, simply by using Hölder's inequality.

Consider the term

$$\rho_i \, |u^{(i)}|^2 = \rho_i \, |u^{(i)}|^{\frac{1}{\gamma}} |u^{(i)}|^{2-\frac{1}{\gamma}} \,.$$

According to Proposition 4.3 or 4.4, respectively, we know that $\rho_i|u^{(i)}|^{\frac{1}{\gamma}} \in L^\gamma_{loc}(\Omega)$. Moreover, $|u^{(i)}|^{2-\frac{1}{\gamma}} \in L^{\frac{6}{2-\frac{1}{\gamma}}}(\Omega)$ since we imposed that $u^{(i)} \in H^1(\Omega;\mathbb{R}^3)$. From these two inclusions we obtain by Hölder's inequality

$$|\rho_i| \, |u^{(i)}|^2 \in L^{\frac{6\gamma}{5+2\gamma}}_{loc}(\Omega) \,.$$

The estimate for ρ_i then follows from the momentum equation (4.2):

We write equation (4.2) in a symbolized form to underline the idea of the proof:

$$\sum_k D(\nabla u^{(k)}) + D(\rho_i |u^{(i)}|^2) = D(P_i(\rho)) + \text{ lower-order terms}\,, \tag{4.15}$$

where D denotes various first-order derivatives.

We have just shown that the term $\rho_i |u^{(i)}|^2 \in L^{\frac{6\gamma}{5+2\gamma}}(\Omega)$ locally. According to the classical L^p-theory (cf. [MNRR96, Theorem 1.14], [Neč66]), it follows from equation (4.15) that $P_i(\rho) \in L_{loc}^{\frac{6\gamma}{5+2\gamma}}(\Omega)$, i.e. $\rho_i \in L_{loc}^{\frac{6\gamma^2}{5+2\gamma}}(\Omega)$, as long as $\frac{6\gamma}{5+2\gamma} \leq 2$, which is the case for $\gamma \leq 5$.

Thus, the proof of Theorem 4.1 and 4.2 has been completed.

Chapter 5

An exponential estimate for the Stokes-like system for mixtures

In this chapter we present a method which delivers even more regularity for the solutions of the Stokes-like problem than proved in Chapter 2, Section 2. Like in the previous chapter on L^p-estimates for the steady mixture model, the results presented here are only regularity, but not existence results.

As in Chapter 2, we consider the Stokes problem

$$\operatorname{div}\left(\rho_i u^{(i)}\right) = 0\,, \tag{5.1}$$

$$\sum_{k=1}^{2} L_{ik} u^{(k)} = -\nabla P_i(\rho) + \rho_i f^{(i)} + I^{(i)}, \tag{5.2}$$

$\rho_i \geq 0, i = 1, 2$, in a bounded domain $\Omega \subset \mathbb{R}^3$. The operators L_{ik} are given by

$$L_{ik} = -\mu_{ik}\Delta - (\lambda_{ik} + \mu_{ik})\nabla \operatorname{div}$$

and are assumed to fulfill the following ellipticity condition for a constant $c_0 > 0$

$$\sum_{i,k=1}^{2} \int_\Omega L_{ik} u^{(k)} \cdot u^{(i)}\, dx \geq c_0 \int_\Omega |\nabla u|^2\, dx\,. \tag{5.3}$$

Furthermore, the equations (5.1)–(5.2) are complemented by

$$\int_\Omega \rho_i\, dx = 1\,. \tag{5.4}$$

We impose no-slip boundary conditions for the velocity fields:

$$u^{(i)}\big|_{\partial\Omega} = 0\,. \tag{5.5}$$

In contrast to Chapter 2, we deal with a pressure law which is nonlinear and behaves like $|\rho|^\gamma$ for a $\gamma \geq 2$. We assume that for a $\gamma \geq 2$ the pressure $P(\rho) = (P_1(\rho), P_2(\rho))^T$ fulfills the following *growth condition*: there is $K_1 > 0$ such that for all $\rho = (\rho_1, \rho_2)^T$ and $\hat{\rho} = (\hat{\rho}_1, \hat{\rho}_2)^T$

$$|P(\rho) - P(\hat{\rho})| \leq K_1 \left(|\rho|^{\gamma-1} + |\hat{\rho}|^{\gamma-1}\right) |\rho - \hat{\rho}| \, . \tag{5.6}$$

Moreover, the pressure law is assumed to fulfill the coerciveness condition

$$(\rho_1^m, \rho_2^m)\, A_0 \left(\begin{array}{c} P_1(\rho) \\ P_2(\rho) \end{array} \right) \geq C_0 |\rho|^{\gamma+m} - K \tag{5.7}$$

for all $\rho_1, \rho_2 \geq 0$ and for $m \geq 1$ as in Chapter 2, Section 2, where we proved some regularity results for the Stokes-like system. We obtained that $\rho_i \in L^p_{loc}(\Omega)$ and $\nabla u^{(i)} \in L^p_{loc}(\Omega; \mathbb{R}^{3\times 3})$ for all p with $1 \leq p < \infty$. As was remarked in Chapter 2.2, the regularity proof works not only for a linear pressure but also for other pressure laws behaving like $|\rho|^\gamma$ as long as a coerciveness condition like (5.7) is satisfied.

Now, we want to obtain the following local estimate:

$$e^{|\rho|^{\frac{\gamma}{2}}} \in L^1_{loc}(\Omega) \, . \tag{5.8}$$

In the quasi-stationary case

$$\begin{aligned} (\rho_i)_t + \operatorname{div}\left(\rho_i u^{(i)}\right) &= 0 \, , \\ \sum_{k=1}^{2} L_{ik} u^{(k)} &= -\nabla P_i(\rho) + \rho_i f^{(i)} + I^{(i)} \, , \end{aligned}$$

which is considered by Frehse and Weigant in [FW04], the authors are able to obtain an L^∞-estimate for the densities. In fact, they even prove the Lipschitz-continuity of the density functions. With the help of the L^∞-estimate for ρ_i they can also estimate the gradient of the densities.

It would be very interesting and is still open to prove an L^∞-estimate for the densities also for the Stokes-like system. The exponential estimate which we derive in this chapter has to be seen as a step in this direction.

We prove the following

Theorem 5.1 *Let $(\rho, u), \rho = (\rho_1, \rho_2)^T, u = (u^{(1),T}, u^{(2),T})^T$, be a solution to (5.1)–(5.2), (5.4), (5.5) with the pressure P fulfilling the conditions (5.6) and (5.7) with $\rho_i \in L^\gamma(\Omega), u^{(i)} \in H^1_0(\Omega; \mathbb{R}^3)$, $i = 1, 2$. Then the following estimate holds:*

$$e^{|\rho|^{\frac{\gamma}{2}}} \in L^1_{loc}(\Omega) \, . \tag{5.9}$$

Proof:
To prove Theorem 5.1, we use again the equation for the effective viscous flux.

We have for $i = 1, 2$ as derived in Chapter 2, Section 2 the following equation:

$$\int_\Omega \hat{\beta}_i \operatorname{div} u^{(i)} \Delta\left(\tau_m \psi^{(i)}\right) dx = \int_\Omega (A_0 P(\rho))_i \, \Delta\left(\tau_m \psi^{(i)}\right) dx - \int_\Omega \left(A_0 \operatorname{div} \Delta^{-1}(\rho f + I)\right)_i \Delta\left(\tau_m \psi^{(i)}\right) dx\,. \tag{5.10}$$

This time, we solve

$$\Delta \psi^{(i)} = \rho_i^{\frac{\gamma}{2}m} \tau_m \tag{5.11}$$

in $\mathbb{R}^3$, where the functions under consideration are extended by zero outside the domain Ω.

The extended functions $\rho_i, u^{(i)}$ solve the continuity equation (5.1) in $\mathcal{D}'(\mathbb{R}^3)$, cf. [NN02, Lemma 2.1].

The localization function τ_m is chosen in such a way that $\tau_m = 1$ in the ball $B_{r_{m+1}}$ and $\tau_m = 0$ outside of B_{r_m}, where for every $m \in \mathbb{N}$

$$r_m = 2 - \frac{\sum_{j=1}^{m} \frac{1}{j^{1+\delta}} - 1}{\sum_{j=1}^{\infty} \frac{1}{j^{1+\delta}} - 1}$$

with $\delta > 0$ small.
With this definition we have $r_1 = 2$ and in the limit $r_m \to 1$ as $m \to \infty$. The domain Ω is chosen in such a way that $B_2 \subset \Omega$ and $B_1 \subset\subset \Omega$ (we can use an appropriate scaling and translation).

The gradient of τ_m can be estimated by

$$\begin{aligned} |\nabla \tau_m| &\le \left| \frac{1}{r_m - r_{m+1}} \right| = \left(\sum_{j=1}^{\infty} \frac{1}{j^{1+\delta}} - 1 \right) (m+1)^{1+\delta} \\ &\le K(m+1)^{1+\delta}\,. \end{aligned}$$

We will use the regularity properties for ρ_i and $\nabla u^{(i)}$, which we have proved in Chapter 2.2 of this thesis:

$$\begin{aligned} &\rho_i \in L^p_{loc}(\Omega) \text{ for all } 1 \le p < \infty\,, \\ &\nabla u^{(i)} \in L^p_{loc}(\Omega; \mathbb{R}^{3\times 3}) \text{ for all } 1 \le p < \infty\,. \end{aligned}$$

In particular, we will use that $u^{(i)} \in L^\infty(\Omega; \mathbb{R}^3)$, which follows from the embedding theorems.

With the above choice of $\psi^{(i)}$ in equation (5.10) we obtain

$$
\begin{aligned}
&\int_\Omega (A_0P(\rho))_i\, \rho_i^{\frac{\gamma}{2}m} \tau_m^2\, dx \\
&= \int_\Omega \hat\beta_i \operatorname{div} u^{(i)} \rho_i^{\frac{\gamma}{2}m}\tau_m^2 + 2\int_\Omega \hat\beta_i \operatorname{div} u^{(i)} \nabla\tau_m\cdot\nabla\Delta^{-1}\left(\rho_i^{\frac{\gamma}{2}m}\tau_m\right)\, dx \\
&+ \int_\Omega \hat\beta_i \operatorname{div} u^{(i)} \Delta\tau_m\Delta^{-1}\left(\rho_i^{\frac{\gamma}{2}m}\tau_m\right)\, dx - 2\int_\Omega (A_0P(\rho))_i\, \nabla\tau_m\cdot\nabla\Delta^{-1}\left(\rho_i^{\frac{\gamma}{2}m}\tau_m\right)\, dx \\
&- \int_\Omega (A_0P(\rho))_i\, \Delta\tau_m\Delta^{-1}\left(\rho_i^{\frac{\gamma}{2}m}\tau_m\right)\, dx \\
&+ \int_\Omega \left(A_0 \operatorname{div}\Delta^{-1}(\rho f + I)\right)_i \Delta\left(\tau_m\Delta^{-1}\left(\rho_i^{\frac{\gamma}{2}m}\tau_m\right)\right)\, dx\,. \qquad (5.12)
\end{aligned}
$$

The term on the left-hand side of (5.12) is what we want to estimate (using the coerciveness condition); we consider now the integrals on the right-hand side:
The first integral gives with the aid of integration by parts

$$
\begin{aligned}
&\int_\Omega \hat\beta_i \operatorname{div} u^{(i)} \rho_i^{\frac{\gamma}{2}m}\tau_m^2\, dx \\
&= -\frac{\gamma}{2}m\int_\Omega \hat\beta_i u^{(i)}\cdot\nabla\rho_i\rho_i^{\frac{\gamma}{2}m-1}\tau_m^2\, dx - \int_\Omega \hat\beta_i u^{(i)}\cdot\nabla\tau_m^2\rho_i^{\frac{\gamma}{2}m}\, dx \\
&= -\frac{\frac{\gamma}{2}m}{\frac{\gamma}{2}m-1}\int_\Omega \hat\beta_i\rho_i u^{(i)}\cdot\nabla\rho_i^{\frac{\gamma}{2}m-1}\tau_m^2\, dx - \int_\Omega \hat\beta_i u^{(i)}\cdot\nabla\tau_m^2\rho_i^{\frac{\gamma}{2}m}\, dx \\
&= \frac{\frac{\gamma}{2}m}{\frac{\gamma}{2}m-1}\int_\Omega \hat\beta_i\rho_i^{\frac{\gamma}{2}m} u^{(i)}\cdot\nabla\tau_m^2\, dx - \int_\Omega \hat\beta_i u^{(i)}\cdot\nabla\tau_m^2\rho_i^{\frac{\gamma}{2}m}\, dx \\
&= \frac{1}{\frac{\gamma}{2}m-1}\int_\Omega \hat\beta_i\rho_i^{\frac{\gamma}{2}m} u^{(i)}\cdot\nabla\tau_m^2\, dx
\end{aligned}
$$

using (5.1). This integral is estimated as follows using $u^{(i)} \in L^\infty(\Omega;\mathbb{R}^3)$ and the properties of $\nabla\tau_m$:

$$
\left|\frac{1}{\frac{\gamma}{2}m-1}\int_\Omega \hat\beta_i\rho_i^{\frac{\gamma}{2}m} u^{(i)}\cdot\nabla\tau_m^2\, dx\right| \le \frac{K}{\frac{\gamma}{2}m-1}(m+1)^{1+\delta}\int_{B_{r_m}} \rho_i^{\frac{\gamma}{2}m}\, dx\,.
$$

The next integral in (5.12) gives

$$
\begin{aligned}
&\left|\int_\Omega \hat\beta_i \operatorname{div} u^{(i)}\nabla\tau_m\cdot\nabla\Delta^{-1}\left(\rho_i^{\frac{\gamma}{2}m}\tau_m\right)\, dx\right| \\
&\le \|\nabla u^{(i)}\|_{L^p_{loc}}(m+1)^{1+\delta}\|\nabla\Delta^{-1}\left(\rho_i^{\frac{\gamma}{2}m}\tau_m\right)\|_{L^{\frac{3}{2}-\delta'}(B_{r_m})}\cdot\|1\|_{L^{3+\delta''}(B_{r_m}\setminus B_{r_{m+1}})} \\
&\quad \text{for a large } p \text{ and suitable } \delta'>0, \delta''>0\,, \text{ using Hölder's inequality,} \\
&\le (m+1)^{1+\delta}\left((m+1)^{1+\delta}\right)^{-\frac{1}{3+\delta''}} K\int_{B_{r_m}} \rho_i^{\frac{\gamma}{2}m}\, dx
\end{aligned}
$$

using $\|\nabla\Delta^{-1}(\rho_i^{\frac{\gamma}{2}m}\tau_m)\|_{L^{\frac{3}{2}-\delta'}(B_{r_m})} \leq \|\rho_i^{\frac{\gamma}{2}m}\tau_m\|_{L^1(B_{r_m})}$ and the measure of the annulus. Here, we have also used that $\nabla u^{(i)} \in L^p_{loc}(\Omega;\mathbb{R}^{3\times 3})$ for all $1 \leq p < \infty$.

Moreover, we obtain from (5.12)

$$\begin{aligned}
&\left|\int_\Omega \hat\beta_i \operatorname{div} u^{(i)} \Delta\tau_m \Delta^{-1}\left(\rho_i^{\frac{\gamma}{2}m}\tau_m\right)\, dx\right| \\
&\leq K\|\nabla u^{(i)}\|_{L^p_{loc}} \left((m+1)^{1+\delta}\right)^2 \|\Delta^{-1}\left(\rho_i^{\frac{\gamma}{2}m}\tau_m\right)\|_{L^{3-\delta'}(B_{r_m})}\|1\|_{L^{\frac{3}{2}+\delta''}(B_{r_m}\setminus B_{r_{m+1}})} \\
&\leq K\left((m+1)^{1+\delta}\right)^2 \left((m+1)^{1+\delta}\right)^{-\frac{1}{\frac{3}{2}+\delta''}} \int_{B_{r_m}} \rho_i^{\frac{\gamma}{2}m}\, dx
\end{aligned}$$

for suitable $\delta' > 0, \delta'' > 0$ and p large,
using $\|\Delta^{-1}\left(\rho_i^{\frac{\gamma}{2}m}\tau_m\right)\|_{L^{3-\delta'}(B_{r_m})} \leq \|\rho_i^{\frac{\gamma}{2}m}\tau_m\|_{L^1(B_{r_m})}$, $\|\nabla u^{(i)}\|_{L^p_{loc}} \leq K$ and
meas $(B_{r_m} \setminus B_{r_{m+1}}) = \left((m+1)^{1+\delta}\right)^{-1}$.

The two integrals with $P(\rho)$ are treated analogously using $\rho_i \in L^p_{loc}(\Omega)$ for all $1 \leq p < \infty$. The terms with f and I are lower-order terms and even simpler to treat.

Summing over i and using the coerciveness condition (5.7), we obtain from (5.12)

$$\int_{B_{r_{m+1}}} |\rho|^{\frac{\gamma}{2}(m+2)}\, dx = \int_{B_{r_{m+1}}} |\rho|^{\gamma+\frac{\gamma}{2}m}\, dx \leq K\left((m+1)^{1+\delta}\right)^{\frac{4}{3}+\tilde\delta} \int_{B_{r_m}} |\rho|^{\frac{\gamma}{2}m}\, dx$$

for a $\tilde\delta > 0$. Dividing by $(m+2)!$ leads to

$$\int_{B_{r_{m+1}}} \frac{|\rho|^{\frac{\gamma}{2}(m+2)}}{(m+2)!}\, dx \leq K\frac{\left((m+1)^{1+\delta}\right)^{\frac{4}{3}+\tilde\delta}}{(m+2)(m+1)} \int_{B_{r_m}} \frac{|\rho|^{\frac{\gamma}{2}m}}{m!}\, dx\,.$$

For m_0 large enough the fraction

$$\frac{\left((m_0+1)^{1+\delta}\right)^{\frac{4}{3}+\tilde\delta}}{(m_0+2)(m_0+1)} =: \varepsilon$$

is small, and for $m \geq m_0$ the quantity

$$\frac{\left((m+1)^{1+\delta}\right)^{\frac{4}{3}+\tilde\delta}}{(m+2)(m+1)}$$

becomes even smaller.

Now, we sum over m from m_0 to ∞ and add also the rest of the series from 1 to m_0, and we get

$$\sum_{m=1}^{\infty}\int_{B_{r_{m+1}}} \frac{|\rho|^{\frac{\gamma}{2}m}}{m!}\, dx \leq \varepsilon\sum_{m=1}^{\infty}\int_{B_{r_m}} \frac{|\rho|^{\frac{\gamma}{2}m}}{m!}\, dx\,, \tag{5.13}$$

which gives the desired result:

$$\int_{B_{r_\infty}} e^{|\rho|^{\frac{\gamma}{2}}}\, dx \leq K$$

for $B_{r_\infty} = B_1$ the ball with radius 1, which is assumed to lie within the domain Ω.

So, in particular, for $\gamma > 2$, we have an estimate for $e^{|\rho|^{1+\varepsilon'}}$ in $L^1_{loc}(\Omega)$.

$\square$

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