Cassinian Ovals*

Level function in 3DXM:

\[ f(x, y) := (x - aa)^2 + y^2 \cdot ((x + aa)^2 + y^2) - bb^4 \]

The default \textit{Color Morph} varies \( bb = ff^{1/4} \) instead of \( ff \).

The Cassinian Ovals (or Ovals of Cassini) were first studied in 1680 by Giovanni Domenico Cassini (1625–1712, aka Jean-Dominique Cassini) as a model for the orbit of the Sun around the Earth.

A Cassinian Oval is a plane curve that is the locus of all points \( P \) such that the \textit{product of the distances} of \( P \) from two fixed points \( F_1, F_2 \) has some constant value \( c \), or

\[ \frac{PF_1}{PF_2} = c. \]

Note the analogy with the definition of an ellipse (where product is replaced by sum). As with the ellipse, the two points \( F_1 \) and \( F_2 \) are called \textit{foci} of the oval. If the origin of our coordinates is the midpoint of the two foci and the \( x \)-axis the line joining them, then the foci will have the coordinates \((a, 0)\) and \((-a, 0)\). Following convention, \( b := \sqrt{c}. \) Then the condition for a point \( P = (x, y) \) to lie on the oval becomes: \((x - a)^2 + y^2)^{1/2}((x + a)^2 + y^2)^{1/2} = b^2. \)

Squaring both sides gives the following \textit{quartic polynomial equation} for the Cassinian Oval:

\[ ((x - a)^2 + y^2)((x + a)^2 + y^2) = b^4. \]

When \( b \) is less that half the distance \( 2a \) between the foci, i.e., \( b/a < 1 \), there are two branches of the curve. When

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* This file is from the 3D-XplorMath project. Please see:

http://3D-XplorMath.org/
The following image shows a family of Cassinian Ovals with \( a = 1 \) and several different values of \( b \).

In 3D-XplorMath, you can change the value of parameter \( b = bb \) in the Settings Menu \( \rightarrow \) SetParameters. An animation of varying values of \( b \) can be seen from the Animate Menu \( \rightarrow \) Color Morph.

Bipolar equation: \( r_1 r_2 = b^2 \)

Polar equation: \( r^4 + a^4 - 2r^2 a^2 \cos(2\theta) = b^4 \)

A parametrization for Cassini’s oval is \( r(t) \cdot (\cos(t), \sin(t)) \),

\[
r^2(t) := a^2 \cos(2t) + \sqrt{(-a^4 + b^4) + a^4(\cos(2t))^2},
\]

\( t \in (0, 2\pi] \), and \( a < b \). This parametrization only generates parts of the curve when \( a > b \).
By default 3D-XplorMath shows how the product definition of the Cassinian ovals leads to a *ruler and circle* construction based on the following circle theorem about products of segments:

\[ CD : CE = CB : CF \quad \rightarrow \quad CD \cdot CF = CB \cdot CE \]

**Cassianian Ovals as sections of a Torus**

Let \( c \) be the radius of the generating circle and \( d \) the distance from the center of the tube to the directrix of the torus. The intersection of a plane \( c \) distant from the torus’ directrix is a Cassinian oval, with \( a = d \) and \( b^2 = \sqrt{4cd} \), where \( a \) is half of the distance between foci, and \( b^2 \) is the constant product of distances.

Cassianian ovals with a large value of \( b^2 \) approach a circle, and the corresponding torus is one such that the tube radius is larger than the center to directrix, that is, a self-intersecting torus without the hole. This surface also approaches a sphere.
Note that the two tori in the figure below are not identical. Arbitrary vertical slices of a torus are called Spiric Sections. In general they are *not* Cassinian ovals.
**Proof:** Start with the equation of a torus

\[
(\sqrt{x^2 + y^2} - d)^2 + z^2 = c^2.
\]

Insert \( y = c \), rearrange and square again:

\[
x^2 + z^2 + d^2 = 2d\sqrt{x^2 + c^2}, \quad (x^2 + z^2 + d^2)^2 = 4d^2(x^2 + c^2).
\]

Now multiply the factors of the implicit equation of an Cassinian oval and rearrange

\[
((x - a)^2 + y^2) \cdot ((x + a)^2 + y^2) = b^4,
\]

\[
(x^2 - a^2)^2 + y^4 + 2y^2(x^2 + a^2) = b^4,
\]

\[
(x^2 + y^2)^2 + 2a^2(y^2 - x^2) = b^4 - a^4.
\]

These two equations match because of \( a = d, \ b^2 = 2dc \), after rotation of the y-axis into the z-axis.

Curves that are the locus of points the product of whose distances from n points is constant are discussed on pages 60–63 of Visual Complex Analysis by Tristan Needham. XL.